Asymptotic Expansions: A Practical Guide

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1 Definitions

Definition 1.1. *Let* $x_0 \in \bar{\mathbb{R}}$. *A sequence of numerical functions* $(\phi_n)_{n \geqslant 1}$ *is a sequence of gauge functions as* $x \to x_0$ *if, for all* $n \geqslant 1$, $\phi_{n+1}(x) = o(\phi_n(x))$ *as* $x \to x_0$.

Definition 1.2. *Consider a sequence of gauge functions* $(\phi_k)_{k \geqslant 1}$ *as* $x \to x_0$. *The sum* $\sum_{k=1}^{n} a_k \phi_k$ *is an asymptotic expansion (of order n) of* f, *as* $x \to x_0$ *if*

$$f(x) - \sum_{k=1}^{n} a_k \phi_k(x) = o(\phi_n(x)),$$

and we write $f(x) \sim \sum_{k=1}^{n} a_k \phi_k(x)$ *as* $x \to x_0$.
The series $\sum_{k=1}^{\infty} a_k \phi_k$ *is an asymptotic expansion (or asymptotic series) of* f *as* $x \to x_0$, *if for all* $n \geqslant 1$,

$$f(x) - \sum_{k=1}^{n} a_k \phi_k(x) = o(\phi_n(x))$$

and we write $f(x) \sim \sum_{k=1}^{\infty} a_k \phi_k(x)$ *as* $x \to x_0$.

Observation: In practice, the series $\sum_{k=1}^{\infty} a_k \phi_k(x)$ may diverge for all $x \neq x_0$. We will see several examples of this in these notes.

2 Techniques

In this section we describe some general techniques to obtain asymptotic expansions.

2.1 Integration by parts

We recall the integration by parts formula which is, as we will see with examples, the main technique to obtain asymptotic expansions.

Proposition 2.1. *Let* f *and* g *be continuously differentiable with integrable derivative on the open interval* (a, b). *Then,*

$$\int_a^b f'(t) g(t) dt = [f(t) g(t)]_a^b - \int_a^b f(t) g'(t) dt$$

where the term in square brackets is treated as $\lim_{t \to b^-} - \lim_{t \to a^+}$.

2.2 Taylor's formula

Iterating integration by parts, we can prove the very useful Taylor formula with integral remainder.

Theorem 2.1. *Let $f \in \mathcal{C}^n(\mathbb{R})$. Then for all pair of reals a and x we have*

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^x f^{(n)}(t)\frac{(x-t)^{n-1}}{(n-1)!}dt.$$

Proof. We start with the fundamental identity of calculus

$$f(x) = f(a) + \int_a^x f'(t)dt.$$

Then, we apply repeated integration by parts to the integral term. This leads to the desired result. □

Corollary 2.1. *(Taylor formula with Lagrange remainder) Under the above conditions, there exists a real θ which belongs to the interval limited by a and x such that*

$$R_n(a,x) := \int_a^x f^{(n)}(t)\frac{(x-t)^{n-1}}{(n-1)!}dt = \frac{f^{(n)}(\theta)}{n!}(x-a)^n.$$

Proof. This is just an application of the second mean value theorem to $R_n(a,x)$. □

2.3 Laplace's method

Laplace's method is a very general technique for obtaining the asymptotic behaviour as $x \to \infty$ of integrals of the form

$$F(x) = \int_a^b f(t)e^{x\phi(t)}dt$$

where $-\infty \leqslant a < b \leqslant \infty$. We assume here that f and ϕ are real continuous function. The case f complex can also be treated by considering separately its real and imaginary parts.

The basic idea is the following: if the real continuous function ϕ has its maximum on the interval $[a,b]$ at $t = c \in \mathbb{R}$ and if $f(c) \neq 0$, then it is only the immediate neighbourhood of $t = c$ that contributes to the full asymptotic expansion of F for large x. That is, we may approximate the integral $F(x)$ by $F(x;\varepsilon)$, where

$$F(x;\varepsilon) = \int_{c-\varepsilon}^{c+\varepsilon} f(t)e^{x\phi(t)}dt$$

if $a < c < b$,

$$F(x;\varepsilon) = \int_{a}^{a+\varepsilon} f(t)e^{x\phi(t)}dt$$

if the maximum of $\phi(t)$ occurs at $t = a$, and

$$F(x;\varepsilon) = \int_{b-\varepsilon}^{b} f(t)e^{x\phi(t)}dt$$

if the maximum of $\phi(t)$ occurs at $t = b$. These results are true since in each case, the other contribution of the integral is exponentially small compared to $F(x)$ as $x \to \infty$. The truncation of $F(x)$ using $F(x;\varepsilon)$ is helpful since $\varepsilon > 0$ may be chosen so small that is valid to replace $f(t)$ and $\phi(t)$ by their Taylor series about $t = c$.

Following this idea, when $\phi(t) = t$, we have the following rigorous result which allows x to be complex.

Lemma 2.1 (Watson's Lemma). *Let $0 < T \leqslant \infty$ and f be a complex valued function of a real variable t such that:*
a) f is continuous on $(0,T)$;
b)

$$f(t) \sim \sum_{n=0}^{\infty} a_n t^{\lambda_n - 1}, \text{ as } t \to 0, \text{ with } 0 < \lambda_0 < \lambda_1 < \ldots;$$

c) in the case $T < \infty$, $\int_0^T |f(t)|dt < \infty$ and in the case $T = \infty$, for some fixed $c > 0$,

$$f(t) = O(e^{ct}), \text{ as } t \to \infty.$$

Then, for all $0 < \delta < \pi/2$, we have

$$F(z) := \int_0^T e^{-zt} f(t)dt \sim \sum_{n=0}^{\infty} a_n \frac{\Gamma(\lambda_n)}{z^{\lambda_n}}$$

as $|z| \to \infty$ and $|\arg(z)| \leqslant \frac{\pi}{2} - \delta$.

Proof. We only prove the case $T = \infty$. First, observe that $F(z)$ is well defined when $\mathrm{Re}(z) > c$ if f satisfies the three conditions of the lemma. Now, note that b) implies that

$$\Big|f(t) - \sum_{k=0}^{n-1} a_k t^{\lambda_k - 1}\Big| \leqslant M t^{\lambda_n - 1}, \text{ as } t \to 0,$$

where $M > 0$ is some constant. Together with c) this implies that

$$\Big|f(t) - \sum_{k=0}^{n-1} a_k t^{\lambda_k - 1}\Big| \leqslant K e^{ct} t^{\lambda_n - 1}, \text{ for } t > 0,$$

where $K > 0$ is some constant. Hence we have

$$\Big|F(z) - \sum_{k=0}^{n-1} a_k \int_0^\infty e^{-zt} t^{\lambda_k - 1} dt\Big| \leqslant K \int_0^\infty e^{-(\mathrm{Re}(z) - c)t} t^{\lambda_n - 1} dt.$$

Note that we have for $\mathrm{Re}(z) > c$ (since $G_k(z) := \int_0^\infty e^{-zt} t^{\lambda_k - 1} dt$ is holomorphic on $\{z \in \mathbb{C} : \mathrm{Re}(z) > c\}$),

$$\int_0^\infty e^{-zt} t^{\lambda_k - 1} dt = \frac{1}{z^{\lambda_n}} \int_0^\infty e^{-u} u^{\lambda_n - 1} du = \frac{\Gamma(\lambda_n)}{z^{\lambda_n}}.$$

Hence we have for $\mathrm{Re}(z) > c$,

$$\Big|F(z) - \sum_{k=0}^{n-1} a_k \frac{\Gamma(\lambda_n)}{z^{\lambda_n}}\Big| \leqslant K \frac{\Gamma(\lambda_n)}{(\mathrm{Re}(z) - c)^{\lambda_n}} = K \frac{\Gamma(\lambda_n)}{|z|^{\lambda_n}} \Big(\frac{|z|}{\mathrm{Re}(z) - c}\Big).$$

Since $|\arg(z)| \leqslant \frac{\pi}{2} - \delta$, we have that $\mathrm{Re}(z) \geqslant |z| \sin\delta$ which implies that $\mathrm{Re}(z) - c \geqslant \frac{1}{2}|z| \sin\delta$ for $|z|$ large enough. This implies that we have

$$F(z) - \sum_{k=0}^{n-1} a_k \frac{\Gamma(\lambda_n)}{z^{\lambda_n}} = O\big(z^{-\lambda_n}\big),$$

which proves Watson's Lemma. $\square$

Exercise: Using Watson's Lemma, obtain the asymptotic series of

$$\int_0^2 \frac{e^{-xt}}{1+t} dt,$$

as $x \to \infty$.

Watson's Lemma only applies to Laplace integrals $F(x)$ where $\phi(t) = -t$. For more general $\phi(t)$, we can first try the change of variables $s = \phi(t)$ and try to use Watson's Lemma. Sometimes $\phi(t)$ is too complicated for the last change of variables to be useful. In this case, we can adopt a more direct approach. In the following, we present a non exhaustive list of cases for which we can obtain the leading terms in the asymptotic expansions.

Case 1: Suppose that ϕ has a global maximum on (a, b) at a point c and $f(c) \neq 0$. In this case, we have

$$F(x) \sim \Big(\frac{2\pi}{-x\phi''(c)}\Big)^{1/2} f(c)e^{x\phi(c)}, \text{ as } x \to \infty. \tag{1}$$

Case 2: Suppose that ϕ has a global maximum at point a, $\phi'(a) < 0$ and $f(a) \neq 0$. Then,

$$F(x) \sim -\frac{f(a)e^{x\phi(a)}}{x\phi'(a)}, \text{ as } x \to \infty.$$

Case 3: Suppose that ϕ has a global maximum at point a, $\phi'(a) = 0$ and $f(a) \neq 0$. Then,

$$F(x) \sim \frac{1}{2}\Big(\frac{2\pi}{-x\phi''(a)}\Big)^{1/2} f(a)e^{x\phi(a)}, \text{ as } x \to \infty.$$

Case 4: Suppose that ϕ has a global maximum on (a, b) at a point c, $\phi^{(j)}(c) = 0$, for $j < p$, $\phi^{(p)}(c) < 0$ and $f(c) \neq 0$. In this case, we have

$$F(x) \sim 2\Gamma(1 + 1/p)\Big(\frac{p!}{-x\phi^{(p)}(c)}\Big)^{1/p} f(c)e^{x\phi(c)}, \text{ as } x \to \infty.$$

Each one of the above expressions can be obtained using the following three-step method:
1. Approximate $F(x)$ by $F(x; \varepsilon)$.
2. Use Taylor formula for f and ϕ at point c.
3. Compute the integrals by sending $\varepsilon \to \infty$.
The third step is the most difficult to understand, since it may be absurd to

first consider ε small and then send ε to infinity. Nevertheless, it is worth observing that this last step only produce an exponentially small error.

Exercise: Apply the above method to obtain the leading behaviour of

$$\int_0^1 \sin t \, e^{-xt^4} dt$$

as $x \to \infty$.

Considering higher order Taylor expansions for f and ϕ, we can obtain the first correction term for $F(x)$. For example, if $a < c < b$, $\phi''(c) < 0$ and at least one of $f(c)$, $f'(c)$ or $f''(c)$ is different from 0, we can obtain

$$F(x) \sim \Big(\frac{2\pi}{-x\phi''(c)}\Big)^{1/2} e^{x\phi(c)} \Bigg\{ f(c) + \frac{1}{x}\Bigg[-\frac{f''(c)}{2\phi''(c)} + \frac{f(c)\phi^{(4)}(c)}{8[\phi''(c)]^2} + \frac{f'(c)\phi^{(3)}(c)}{2[\phi''(c)]^2} - \frac{5[\phi^{(3)}]^2 f(c)}{24[\phi''(c)]^3} \Bigg]\Bigg\}, \tag{2}$$

as $x \to \infty$. We can go on to obtain higher order terms, but the computations become quickly very fastidious!

Movable maxima

There are two cases where the Laplace method is useful but cannot be applied directly. The first case is when $f(t)$ vanishes exponentially at $t = c$. The second case is when $\sup \phi(t) = \infty$. We consider each of these cases in the following two examples.

Example 1: Let us find the leading behaviour of the following integral

$$F(x) = \int_0^\infty e^{-\frac{1}{t} - xt} dt, \quad \text{as } x \to \infty.$$

Here, $f(t) = e^{-1/t}$ vanishes exponentially as $t \to 0$, which the maximum of $\phi(t) = -t$. We cannot apply Watson's lemma since the asymptotic expansion of $f(t)$ near 0 is null. In order to determine the correct behaviour of $F(x)$, we have to find the location of the true maximum of the full integrand $e^{-\frac{1}{t} - xt}$. This occurs when $t = 1/\sqrt{x}$. Such a maximum is called a movable maximum since its location depends on x. We can now apply Laplace's method if we first transform this movable maximum into a fixed maximum. This can be done by making the change of variables $t = s/\sqrt{x}$. This leads to

$$F(x) = \frac{1}{\sqrt{x}} \int_0^\infty e^{-\sqrt{x}(s+\frac{1}{s})} dt$$

In this form, $f(s) = 1$ and $\phi(s) = s + s^{-1}$ and Laplace's method can be applied directly. The maximum of the new function $\phi(s)$ occurs at $s = 1$ and (1) gives

$$F(x) \sim \sqrt{\pi} \frac{e^{-2\sqrt{x}}}{x^{3/4}}, \quad \text{as } x \to \infty.$$

Example 2: (Stirling's formula for $\Gamma(x)$)
Consider the Gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

We want to obtain the first two leading terms in the asymptotic expansion of $\Gamma(x)$ as $x \to \infty$. Here $f(t) = \frac{e^{-t}}{t}$ and $\phi(t) = \ln t$. Note that $\sup_{t>0} \phi(t) = \infty$, so that the Laplace method is not directly applicable. The supremum of $\phi(t)$ is "reached" when $t \to \infty$ where $f(t)$ is exponentially small. We will find the location of the maximum of $e^{-t} t^x$, neglecting the factor $1/t$ which vanishes algebraically at ∞. The maximum occurs when $t = x$ which is a movable maximum. Doing the change of variables $t = sx$, we obtain

$$\Gamma(x) = x^x \int_0^\infty \frac{e^{-x(s-\ln s)}}{s} ds.$$

Now, $f(s) = 1/s$ and $\phi(s) = -s + \ln s$. The maximum of $\phi(s)$ occurs at $s = 1$ and (1) gives

$$\Gamma(x) \sim x^x e^{-x} \sqrt{\frac{2\pi}{x}}, \quad \text{as } x \to \infty.$$

We can even obtain the next leading term. Applying (2), we obtain

$$\Gamma(x) \sim x^x e^{-x} \sqrt{\frac{2\pi}{x}} \Big(1 + \frac{1}{12x}\Big), \quad \text{as } x \to \infty.$$

2.4 Stationary phase method

In this section, we consider ϕ pure imaginary, that is $\phi = i\psi$, where ψ is a real function. We will study the asymptotic behavior of an integral of the form

$$F(x) = \int_a^b f(t)e^{ix\psi(t)}dt \tag{3}$$

as $x \to \infty$.
To study the behaviour of $F(x)$ we can first try to use integration by parts to develop an asymptotic expansion in inverse power of x.
Example: Consider

$$F(x) = \int_0^1 \frac{e^{ixt}}{1+t}dt.$$

A first integration by parts gives

$$F(x) = -\frac{i}{2x}e^{ix} + \frac{i}{x} - \frac{i}{x}\int_0^1 \frac{e^{ixt}}{(1+t)^2}dt.$$

The last term is actually negligible compared with the boundary terms since it vanishes essentially like $1/x^2$ as $x \to \infty$. To see this, we integrate by parts again

$$\frac{i}{x}\int_0^1 \frac{e^{ixt}}{(1+t)^2}dt = \frac{1}{4x^2}e^{ix} - \frac{1}{x^2} + \frac{2}{x^2}\int_0^1 \frac{e^{ixt}}{(1+t)^3}dt.$$

Then, we observe that

$$\Big|\int_0^1 \frac{e^{ixt}}{(1+t)^3}dt\Big| \leqslant \int_0^1 \frac{1}{(1+t)^3}dt = \frac{3}{8}.$$

Finally we obtain

$$F(x) \sim -\frac{i}{2x}e^{ix} + \frac{i}{x}, \quad \text{as } x \to \infty.$$

Exercise: Iterating integration by parts, obtain the asymptotic series of $F(x)$ as $x \to \infty$.

When integrating by parts, we can often use the following lemma (or some generalised version of it) to show that the integral in the integration by parts formula is negligible in comparison with the boundary term.

Lemma 2.2 (Riemann-Lebesgue lemma)**.** *Let $-\infty \leqslant a < b \leqslant \infty$ and $f \in L^1((a,b))$. We have that*

$$\lim_{x\to\infty} \int_a^b f(t)e^{ixt}dt = 0. \tag{4}$$

Proof. We use the fact $C_c^\infty((a,b))$ is dense in $L^1((a,b))$ and that it is easy to verify (4) for $g \in C_c^\infty((a,b))$ (use integration by parts). □

Exercise: Generalise the Riemann-Lebesgue lemma when ψ is continuously differentiable on $[a,b]$ and $\psi'(t) \neq 0$ for $t \in [a,b]$.

Integration by parts may not work when $\psi'(t) = 0$ for some $t \in [a,b]$. Such a point is called a *stationary* point of ψ. The method of stationary phase gives the leading asymptotic order of such integrals. In the following, we explain the method when $f(a) \neq 0$, $\psi'(a) = 0$ and $\psi'(t) \neq 0$ for $t \in (a,b]$.

$$F(x) = \int_a^{a+\varepsilon} f(t)e^{ix\psi(t)}dt + \int_{a+\varepsilon}^b f(t)e^{ix\psi(t)}dt.$$

The second integral in the above expression vanishes like $1/x$ as $x \to \infty$ because there are no stationary points in the interval $[a+\varepsilon, b]$. We will see that the leading behaviour of $F(x)$ is given by the first integral. To obtain the leading behaviour of the first integral we replace $f(t)$ by $f(a)$ and $\psi(t)$ by $\psi(a) + \psi^{(p)}(a)(t-a)^p$ where $\psi^{(p)}(a) \neq 0$ but $\psi'(a) = \cdots = \psi^{(p-1)}(a) = 0$,

$$F(x) \sim \int_a^{a+\varepsilon} f(a)\exp\Big\{ix\Big[\psi(a) + \frac{\psi^{(p)}(a)}{p!}(t-a)^p\Big]\Big\}dt, \quad \text{as } x \to \infty.$$

Next, we replace ε by ∞, which introduces an error term that vanishes like $1/x$ as $x \to \infty$. Then, making the change of variables $s = t - a$, we have

$$F(x) \sim f(a)e^{ix\psi(a)} \int_0^\infty \exp\Big[ix\frac{\psi^{(p)}(a)}{p!}s^p\Big]ds, \quad \text{as } x \to \infty.$$

Evaluating this last integral (see Appendix, section 4.1), we finally obtain

$$F(x) \sim f(a)e^{ix\psi(a)\pm\frac{i\pi}{2p}}\Big[\frac{p!}{x|\psi^{(p)}(a)|}\Big]^{1/p}\frac{\Gamma(1/p)}{p}, \quad \text{as } x \to \infty,$$

where we use $e^{i\pi/2p}$ if $\psi^{(p)}(a) > 0$ and $e^{-i\pi/2p}$ if $\psi^{(p)}(a) < 0$.

Exercise: Apply the above method to obtain the leading behaviour of

$$\int_0^{\frac{\pi}{2}} e^{ix\cos t} dt$$

as $x \to \infty$.

Higher order asymptotic expansions can be hard to obtain using the stationary phase method because of the contributions of non-stationary points. In this case, we can use the method described in the following section.

2.5 Steepest descent method

The steepest descent method is a general technique for finding the asymptotic behaviour of integrals of the form

$$F(x) = \int_C f(z) e^{xh(z)} dz, \quad \text{as } x \to \infty,$$

where C is a path in the complex z-plane and f and h are holomorphic in some domain of the complex plane that contains C. The basic idea of the method is to deform the path C into a new path C', using Cauchy's theorem, on which h has a constant imaginary part. It happens that C' is also a steepest path (this is a consequence of the Cauchy-Riemann equations), that is, a path where the real part of h have the greatest variations. Then, $F(x)$ may be evaluated asymptotically, as $x \to \infty$, using Laplace's method. Let us see how this works with a first example.

Example 1: We will obtain the asymptotic behaviour of

$$F(x) = \int_0^1 \ln t \; e^{ixt} dt, \quad \text{as } x \to \infty.$$

Before starting, observe that the asymptotic expansion of F cannot be obtained using the stationary phase method (even for the first order term) because there is no stationary point. Also, integration by parts fails here because $\ln 0 = -\infty$!

To obtain an asymptotic expansion of $F(x)$, we first consider the closed path $C' = C + C_1 + C_2 + C_3$ oriented clockwise where $C = \overleftarrow{[0,1]}$, $C_1 = i\overrightarrow{[0,T]}$, $C_2 = \overrightarrow{[0,1]} + iT$ and $C_3 = 1 + i\overleftarrow{[0,T]}$, for some $T > 0$. The path $C_1 = i\overrightarrow{[0,T]}$ is a steepest descent path at point 0 and $C_3 = 1 + i\overrightarrow{[0,T]}$ (pay attention to the orientation) is a steepest descent path at point 1. The path C_2 just makes the connection between C_1 and C_3 and we will see that the contribution due to the integral on C_2 vanishes. By Cauchy's theorem we have that

$$F(x) = \int_{C_1+C_2+C_3} \ln z \; e^{ixz} dz.$$

Then, we let $T \to \infty$. We easily obtain that $\int_{C_2} \ln z \; e^{ixz} dz \to 0$. Then, in the integral along C_1, we set $z = is$ and in the integral along C_3 we set $z = 1+is$, where s is real. Hence, we obtain

$$F(x) = i \int_0^\infty \ln(is) \; e^{-xs} ds - i \int_0^\infty \ln(1+is) \; e^{ix(1+is)} ds. \tag{5}$$

The first integral can be computed explicitly using that $\ln(is) = \ln s + i\frac{\pi}{2}$ (we use the principal branch of the complex logarithm!) and making $u = xs$:

$$\begin{aligned}\int_0^\infty \ln(is) \; e^{-xs} ds &= \frac{1}{x} \int_0^\infty \ln\left(\frac{u}{x}\right) e^{-u} du + i\frac{\pi}{2x} \int_0^\infty e^{-u} du \\ &= -\frac{\ln x}{x} + \frac{1}{x}\Big(-\gamma_E + i\frac{\pi}{2}\Big), \end{aligned} \tag{6}$$

where γ_E is the Euler-Mascheroni constant. The second integral can be estimated asymptotically using Watson's lemma. Using the Taylor series

$$\ln(1+is) = \sum_{n=1}^\infty (-1)^{n+1} \frac{(is)^n}{n},$$

which is valid when $|s| < 1$, we obtain that

$$\int_0^\infty \ln(1+is) \; e^{ix(1+is)} ds \sim -e^{ix} \sum_{n=1}^\infty \frac{(-i)^n (n-1)!}{x^{n+1}}, \quad \text{as } x \to \infty. \tag{7}$$

Finally using (6) and (7) in (5) we deduce that

$$F(x) \sim -\frac{i \ln x}{x} - \frac{i\gamma_E + \pi/2}{x} + ie^{ix} \sum_{n=1}^\infty \frac{(-i)^n (n-1)!}{x^{n+1}}, \quad \text{as } x \to \infty.$$

When z is not a saddle point for h (that is $h(z) \neq 0$) there is a unique steepest path passing through z. In the above example, the function $h(z) = z$

did not have any saddle point and there was a unique steepest path passing through each point. The good choice was to consider the steepest descent paths passing through points 0 and 1 and in the end, the main contribution to the integrals on C_1 and C_3 came from the boundary points 0 and 1 respectively. In the next example, we consider a case where the function h has saddle points and we show how to deal with such points. It is worth mentioning that at saddle points steepest paths can intersect and it is important to choose the right steepest descent path to apply the Laplace method.

Example 2: Consider the Airy function defined by,

$$\mathrm{Ai}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(tx+\frac{t^3}{3})}dt. \tag{8}$$

We will obtain the leading term of $\mathrm{Ai}(x)$, as $x \to \infty$. We first put the above integral in a suitable representation to apply the steepest descent method. Letting $t = x^{1/2}z$, we obtain that

$$\mathrm{Ai}(x) = \frac{x^{1/2}}{2\pi}\int_{-\infty}^{\infty} e^{ix^{3/2}(z+\frac{z^3}{3})}dz =: \frac{x^{1/2}}{2\pi} I(x). \tag{9}$$

Before proceeding, observe that the method of stationary phase does not work here because there is no stationary point! Integration by parts also fails since the boundary terms vanish.

The function $h(z) := i(z + \frac{z^3}{3})$ has saddle points at $z = \pm i$. Now let us write h in term of its real and imaginary parts: $h = \phi + i\psi$, where $z = u + iv$ and

$$\phi(u, v) = -v\Big(1 + u^2 - \frac{1}{3}v^2\Big),$$
$$\psi(u, v) = u\Big(1 + \frac{1}{3}u^2 - v^2\Big).$$

The best choice to obtain an asymptotic expansion of (8) is to find a nice path passing through the saddle point $z = i$. The steepest descent path passing through $z = i$ is given by the equation $\psi(u, v) = \psi(0, 1) = 0$, that is,

$$v = \sqrt{1 + \frac{1}{3}u^2}.$$

There is also a steepest-ascent path passing through i, which one? Then, using Cauchy's theorem we have for $R > 0$,

$$\int_{-R}^{R} e^{x^{3/2}h(z)}dz = \int_{\gamma_1} e^{x^{3/2}h(z)}dz + \int_{\gamma_2} e^{x^{3/2}h(z)}dz + \int_{\gamma_3} e^{x^{3/2}h(z)}dz,$$

where $\gamma_1 = -R + i\overrightarrow{\left[0, \left(1+\frac{1}{3}R^2\right)^{1/2}\right]}$, $\gamma_3 = R + i\overleftarrow{\left[0, \left(1+\frac{1}{3}R^2\right)^{1/2}\right]}$ and

$$\gamma_2 = \left\{z = u + i\left(1+\frac{1}{3}u^2\right)^{1/2}, -R \leqslant u \leqslant R\right\}.$$

Now, letting $R \to \infty$, it is not hard to obtain that the integrals on γ_1 and γ_3 vanish, hence we are left with

$$I(x) = \int_{\tilde{\gamma}_2} e^{x^{3/2}h(z)}dz,$$

where $\tilde{\gamma}_2 = \left\{z = u + i(1+\frac{1}{3}u^2)^{1/2}, u \in \mathbb{R}\right\}$. Let us parametrize the integral on $\tilde{\gamma}_2$. An easy way to do this, is to take $u(s) = \sqrt{3}\sinh s$ and $v(s) = \cosh s$ for $s \in \mathbb{R}$. Then, we find that

$$\int_{\tilde{\gamma}_2} e^{x^{3/2}h(z)}dz = \int_{-\infty}^{\infty}(\sqrt{3}\cosh s + i\sinh s)e^{x^{3/2}\cosh s[2-\frac{8}{3}\cosh^2 s]}ds.$$

The maximum of $g(s) := \cosh s[2-\frac{8}{3}\cosh^2 s]$ occurs at $s = 0$, $g(0) = -2/3$, $g'(0) = 0$ and $g''(0) = -6$. Thus, Laplace's method gives that

$$I(x) = \int_{-\infty}^{\infty}(\sqrt{3}\cosh s + i\sinh s)e^{x^{3/2}\cosh s[2-\frac{8}{3}\cosh^2 s]}ds \sim \frac{\sqrt{\pi}}{x^{3/4}}e^{-\frac{2}{3}x^{3/2}}$$

as $x \to \infty$. Finally, using (9), we deduce that

$$\mathrm{Ai}(x) \sim \frac{1}{2\sqrt{\pi}x^{1/4}}e^{-\frac{2}{3}x^{3/2}}, \quad \text{as } x \to \infty.$$

Actually, in this particular example, using the path that is tangent to the steepest descent curve at $z = i$ we can obtain an asymptotic series for $\mathrm{Ai}(x)$, as $x \to \infty$. Indeed, consider the path $\tilde{\gamma} = \{z = u+i, u \in \mathbb{R}\}$. When $z = u+i$, we have

$$h(z) = -\left(\frac{2}{3}+u^2\right) + i\frac{u^3}{3},$$

so that on $\tilde{\gamma}$ the imaginary part of h is not constant! As we will see, this is not a problem here. By Cauchy's theorem and using the parametrization of $\tilde{\gamma}$, we easily deduce that

$$\begin{aligned} I(x) &= e^{-\frac{2x^{3/2}}{3}}\int_{-\infty}^{\infty} e^{ix^{3/2}\frac{u^3}{3}}e^{-x^{3/2}u^2}du \\ &= 2e^{-\frac{2x^{3/2}}{3}}\int_0^{\infty}\cos\left(x^{3/2}\frac{u^3}{3}\right)e^{-x^{3/2}u^2}du. \end{aligned}$$

Now, making the change of variables $t = xu^2$ in the last integral we obtain that

$$I(x) = \frac{e^{-\frac{2x^{3/2}}{3}}}{\sqrt{x}} \int_0^\infty \frac{\cos(\frac{t^{3/2}}{3})}{\sqrt{t}} e^{-\sqrt{x}t} dt.$$

Applying Watson's lemma we obtain that

$$I(x) \sim \frac{e^{-\frac{2x^{3/2}}{3}}}{x^{3/4}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{\Gamma(3n+1/2)}{(9x^{3/2})^n}, \quad \text{as } x \to \infty$$

and finally using (9)

$$\mathrm{Ai}(x) \sim \frac{e^{-\frac{2x^{3/2}}{3}}}{2\pi x^{1/4}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{\Gamma(3n+1/2)}{(9x^{3/2})^n}, \quad \text{as } x \to \infty.$$

2.6 Euler-Maclaurin summation formulas

Let us first state the classical form of the Euler-Maclaurin summation formula (EMSF).

Theorem 2.2. *Let $m < n$ be two integers and $f \in \mathcal{C}^{2k}([m,n])$, with $k \geqslant 1$. then, we have*

$$\sum_{i=m}^{n} f(i) = \int_m^n f(x)dx + \frac{f(m)+f(n)}{2} + \sum_{j=1}^{k} \frac{b_{2j}}{(2j)!}\big(f^{(2j-1)}(n) - f^{(2j-1)}(m)\big) + R_{2k},$$

where the numbers b_{2j} are the Bernoulli numbers,

$$R_{2k} = -\frac{1}{(2k)!} \int_m^n f^{(2k)}(x) B_{2k}(x - \lfloor x \rfloor) dx$$

and B_{2k} is the $2k$-th Bernoulli polynomial.

For a brief presentation of Bernoulli polynomials and numbers see Appendix, section 4.2.

Observation: To control the error term $|R_{2k}|$ we can use (see Appendix, section 4.2)

$$|R_{2k}| \leqslant \frac{|b_{2k}|}{(2k)!} \int_m^n |f^{(2k)}(x)| dx \leqslant \frac{4}{(2\pi)^{2k}} \int_m^n |f^{(2k)}(x)| dx. \tag{10}$$

Theorem 2.2 can be useful to obtain asymptotic expansions of Riemann sums.

Example 1: Let us consider $S_n = \sum_{k=0}^{n} \frac{1}{1+(\frac{k}{n})^2}$. Applying Theorem 2.2 to $f(x) = \frac{1}{1+(\frac{x}{n})^2}$ together with (10), we easily obtain that

$$S_n = \frac{\pi}{4}n + \frac{3}{4} - \frac{1}{24n} + O\Big(\frac{1}{n^3}\Big), \quad \text{as } n \to \infty.$$

From Theorem 2.2, we can deduce a second form of the EMSF that can be used to obtain asymptotic expansions of $\sum_{i=1}^{n} f(i)$ as $n \to \infty$.

Theorem 2.3. *Let f be a function defined on the interval $[1,\infty)$, $f \in \mathcal{C}^{2k}([1,\infty))$, for some $k \geqslant 1$, and suppose that $f^{(2k)}$ is absolutely integrable. Then, for $n \geqslant 1$,*

$$\sum_{i=1}^{n} f(i) = \int_1^n f(x)dx + \frac{f(n)}{2} + C_f + \sum_{j=1}^{k} \frac{b_{2j}}{(2j)!} f^{(2j-1)}(n) + R'_{2k},$$

where C_f is a constant that only depends on f defined by

$$C_f = \frac{f(1)}{2} - \sum_{j=1}^{k} \frac{b_{2j}}{(2j)!} f^{(2j-1)}(1) - \int_1^{\infty} f^{(2k)}(x) B_{2k}(x - \lfloor x \rfloor)dx$$

and

$$R'_{2k} = \frac{1}{(2k)!} \int_n^{\infty} f^{(2k)}(x) B_{2k}(x - \lfloor x \rfloor)dx.$$

Example 2: (Harmonic series)
Let us obtain an asymptotic expansion of the harmonic series $H_n := \sum_{k=1}^{n} \frac{1}{k}$. Taking $f(x) = 1/x$ in Theorem 2.3 and using the fact that $f^{(m)}(x) = (-1)^m \frac{m!}{x^{m+1}}$, for $m \geqslant 1$, we obtain that

$$H_n = \ln n + C_f + \frac{1}{2n} - \sum_{j=1}^{k} \frac{b_{2j}}{2jn^{2j}} + R'_{2k} \tag{11}$$

for $k \geqslant 1$. Now observe that

$$|R'_{2k}| \leqslant \frac{b_{2k}}{(2k)!} \int_n^{\infty} |f^{(2k)}(x)|dx = \frac{b_{2k}}{2kn^{2k}},$$

that is, R'_{2k} is of the same order (in n) as the last term of the sum in (11). We deduce that

$$H_n \sim \ln n + C_f + \frac{1}{2n} - \sum_{k=1}^{\infty} \frac{b_{2k}}{2kn^{2k}}, \quad \text{as } n \to \infty.$$

Actually, we can even deduce that $C_f = \lim_{n\to\infty}(H_n - \ln n) =: \gamma_E$.

Example 3: (Stirling's formula)
We want to obtain an asymptotic expansion of $\ln n!$ as $n \to \infty$. For this observe that $\ln n! = \sum_{k=1}^{n} \ln k$ and apply Theorem 2.3 to $f(x) = \ln x$. We obtain

$$\ln n! = \left(n + \frac{1}{2}\right) \ln n - n + C_f + \sum_{j=1}^{k} \frac{b_{2j}}{(2j)(2j-1)n^{2j-1}} + R'_{2k},$$

for $k \geqslant 1$. Now, observe that

$$|R'_{2k}| \leqslant \frac{b_{2k}}{(2k)!} \int_n^{\infty} |f^{(2k)}(x)| dx = \frac{b_{2k}}{(2k)(2k-1)n^{2k-1}}.$$

Thus, we deduce that

$$\ln n! \sim \left(n + \frac{1}{2}\right) \ln n - n + C_f + \sum_{k=1}^{\infty} \frac{b_{2k}}{(2k)(2k-1)n^{2k-1}}, \quad \text{as } n \to \infty.$$

Using other techniques, it can be shown that

$$C_f = \lim_{n\to\infty} \left(\ln n! - \left(n + \frac{1}{2}\right) \ln n + n \right) = \ln \sqrt{2\pi}.$$

Exercise: Obtain an asymptotic expansion of $\sum_{k=1}^{n} \sqrt{k}$, as $n \to \infty$. You do not need to explicit C_f.

2.7 Slowly varying functions

Definition 2.1. *Let $a > 0$. A positive function f on $[a, \infty)$ is slowly varying when $x \to \infty$, if for all $t > 0$,*

$$\lim_{x\to\infty} \frac{f(tx)}{f(x)} = 1.$$

We write $f \in \mathcal{SV}$.

Typical examples of slowly varying functions are: $\ln x$, $\ln\ln x$, $\frac{\ln x}{\ln\ln x}$,...

One important result about slowly varying functions is the following

Proposition 2.2. *Let $\rho > -1$. If $f \in \mathcal{SV}$, then*

$$\sum_{j=1}^{n} j^\rho f(j) \sim \frac{n^{\rho+1}}{\rho+1} f(n), \quad \text{as } n \to \infty.$$

Examples:

$$\sum_{j=1}^{n} \frac{\ln j}{\sqrt{j}} \sim 2\sqrt{n}\ln n,$$
$$\sum_{j=3}^{n} \frac{j}{\ln\ln j} \sim \frac{n^2}{2\ln\ln n},$$

as $n \to \infty$.

3 More examples

3.1 Euler integral: part 1

Consider the function defined by the following integral

$$F(x) = \int_0^\infty \frac{e^{-t}}{1+xt} dt$$

for all $x \geqslant 0$. We want to obtain an asymptotic expansion of $F(x)$ as $x \to \infty$.

We first make the change of parameter $\varepsilon = 1/x$ to obtain the integral

$$G(\varepsilon) = \varepsilon \int_0^\infty \frac{e^{-t}}{\varepsilon + t} dt =: \varepsilon H(\varepsilon)$$

We will obtain an asymptotic expansion of $G(\varepsilon)$ as $\varepsilon \to 0$. Now, observe that $H(\varepsilon)$ has a singularity at $\varepsilon = 0$. The idea is to isolate and express the singular part of H with the help of a simple (computable) integral. That is, we rewrite H as follows

$$\begin{aligned} H(\varepsilon) &= \int_0^1 \frac{e^{-t}}{\varepsilon + t} dt + \int_1^\infty \frac{e^{-t}}{\varepsilon + t} dt \\ &= \int_0^1 \frac{e^{-t} - 1}{\varepsilon + t} dt + \int_0^1 \frac{1}{\varepsilon + t} dt + \int_1^\infty \frac{e^{-t}}{\varepsilon + t} dt \\ &= \int_0^1 \frac{e^{-t} - 1}{\varepsilon + t} dt + \ln(1 + \varepsilon) - \ln \varepsilon + \int_1^\infty \frac{e^{-t}}{\varepsilon + t} dt. \end{aligned}$$

This last expression is well suited to obtain an asymptotic expansion of $H(\varepsilon)$, as $\varepsilon \to 0$, since the last two integrals converge as $\varepsilon \to 0$.

Exercise 1: Show that as $x \to \infty$, the asymptotic expansion of F is of the form

$$F(x) \sim \frac{1}{x}(\ln x + a_1 - \frac{\ln x}{x} + \frac{a_2}{x} + \dots).$$

Actually, we can show that $a_1 = \gamma_E$.

Exercise 2: Show that as $x \to 0^+$, the asymptotic expansion of

$$F(x) = \int_0^1 \frac{\ln t}{t + x} dt$$

is of the form

$$F(x) \sim -\frac{1}{2} \ln^2 x - \frac{\pi^2}{6} + x + \dots$$

3.2 Euler integral: part 2

Consider the function defined by the following integral

$$F(x) = \int_0^\infty \frac{e^{-t}}{1 + xt} dt$$

for all $x \geqslant 0$. We want to obtain an asymptotic expansion of $F(x)$ as $x \to 0$. First, we proceed formally. We use the power series expansion

$$\frac{1}{1+xt} = 1 - xt + x^2t^2 - \cdots + (-1)^n x^n t^n + \ldots$$

inside the Euler integral and integrate the result term by term. This gives the Stieltjes series

$$S(x) = 1 - x + 2!x^2 - 3!x^3 + \cdots + (-1)^n n! x^n + \ldots,$$

which diverges for all $x \neq 0$. Now, let us show the following

Proposition 3.1. *For $x \geqslant 0$ and $n \geqslant 0$ we have*

$$|F(x) - \sum_{k=0}^{n} (-1)^k k! x^k| \leqslant (n+1)! x^{n+1}.$$

That is, the Stieltjes series is an asymptotic expansion of $F(x)$ as $x \to 0$.

Proof. Integrating by parts $n+1$ times we obtain that

$$F(x) = \sum_{k=0}^{n} (-1)^k k! x^k + R_{n+1}(x)$$

where

$$R_{n+1}(x) = (-1)^{n+1}(n+1)! x^{n+1} \int_0^\infty \frac{e^{-t}}{(1+xt)^{n+2}} dt.$$

Estimating $|R_{n+1}(x)|$ from above we obtain that

$$|R_{n+1}(x)| \leqslant (n+1)! x^{n+1}$$

which shows the result. □

3.3 Exponential integral E_n

We start with the integral

$$E_1(x) = \int_x^\infty \frac{e^{-t}}{t} dt.$$

We want to obtain a asymptotic expansion of $E_1(x)$ as $x \to \infty$. We use integration by parts technique. We obtain after $n+1$ integrations

$$E_1(x) = \frac{e^{-x}}{x} \sum_{k=0}^{n} (-1)^k \frac{k!}{x^k} + (-1)^{n+1}(n+1)! \int_x^\infty \frac{e^{-t}}{t^{n+2}} dt.$$

Now, we show that the series $\sum_{n=0}^{\infty} (-1)^n \frac{n!}{x^n}$ is an asymptotic expansion of $E_1(x)$ as $x \to \infty$. For this we have

$$|R_{n+1}(x)| = (n+1)! \int_x^\infty \frac{e^{-t}}{t^{n+2}} dt \leqslant (n+1)! \frac{e^{-x}}{x^{n+2}}$$

which shows the desired result.

Now we consider the exponential integral of order $n \geqslant 1$, that is,

$$E_n(x) = \int_x^\infty \frac{e^{-t}}{t^n} dt$$

and obtain an asymptotic expansion for $E_n(x)$ as $x \to \infty$. We first observe that

$$E_{n+1}(x) = \frac{e^{-x}}{nx^n} - \frac{1}{n} E_n(x).$$

From this recurrence relation we find that

$$E_{n+1}(x) = \sum_{k=0}^{n-1} (-1)^k \frac{(n-k-1)!}{n!} \frac{e^{-x}}{x^{n-k}} + \frac{(-1)^n}{n!} E_1(x)$$

Using the asymptotic expansion we just obtained for $E_1(x)$, we deduce, for all $n \geqslant 0$,

$$E_{n+1}(x) \sim \frac{(-1)^n}{n!} \frac{e^{-x}}{x} \sum_{k=n}^{\infty} (-1)^k \frac{k!}{x^k},$$

as $x \to \infty$.

3.4 Incomplete Gamma function

In this section, let us consider the following function, for $a > 0$ and $x \geqslant 0$:

$$\gamma(a, x) := \int_0^x t^{a-1} e^{-t} dt.$$

We want to obtain asymptotic expansions of $\gamma(a,x)$ as $x \to 0$ and $x \to \infty$. We start with the case $x \to 0$. We can use the serie expansion the exponential function. We obtain

$$\gamma(a,x) = \int_0^x t^{a-1}e^{-t}dt = \int_0^x t^{a-1}\sum_{n=0}^{\infty}(-1)^n \frac{t^n}{n!}dt.$$

Interchanging the integral and series (which is perfectly allowed here), we deduce

$$\gamma(a,x) = \int_0^x t^{a-1}\sum_{n=0}^{\infty}(-1)^n \frac{t^n}{n!}dt = x^a \sum_{n=0}^{\infty}(-1)^n \frac{x^n}{(a+n)n!}.$$

This last series converges actually for all x, but for x large the convergence is very slow.

We now treat the case $x \to \infty$. We start by noting that

$$\gamma(a,x) = \Gamma(a) - \int_x^{\infty} t^{a-1}e^{-t}dt =: \Gamma(a) - E_{1-a}(x).$$

We now integrate by parts $E_{1-a}(x)$ successively

$$\begin{aligned}
E_{1-a}(x) &= e^{-x}x^{a-1} + (a-1)\int_x^{\infty} t^{a-2}e^{-t}dt \\
&= \dots \\
&= e^{-x}\Big(x^{a-1} + (a-1)x^{a-2} + \dots + (a-1)\dots(a-n+1)x^{a-n}\Big) \\
&\quad + (a-1)(a-2)\dots(a-n)\int_x^{\infty} t^{a-n-1}e^{-t}dt.
\end{aligned}$$

Note that for $n > a-1$, we have

$$\begin{aligned}
|R_{n+1}(x)| &= \Big|(a-1)(a-2)\dots(a-n)\int_x^{\infty} t^{a-n-1}e^{-t}dt\Big| \\
&= |(a-1)\dots(a-n)|\int_x^{\infty} t^{a-n-1}e^{-t}dt \\
&\leqslant |(a-1)\dots(a-n)|x^{a-n-1}\int_x^{\infty} e^{-t}dt \\
&= |(a-1)\dots(a-n)|x^{a-n-1}e^{-x}.
\end{aligned}$$

This is enough to deduce that

$$E_{1-a}(x) \sim e^{-x}x^a\Big(\frac{1}{x} + \sum_{k=1}^{\infty}(a-1)\dots(a-k)\frac{1}{x^{k+1}}\Big)$$

as $x \to \infty$. Finally we obtain,

$$\gamma(a,x) \sim \Gamma(a) - e^{-x}x^a\Big(\frac{1}{x} + (a-1)\frac{1}{x^2} + (a-1)(a-2)\frac{1}{x^3} + \dots\Big)$$

as $x \to \infty$.

Exercise: Now, we consider the case $a = x$. Show that

$$\gamma(x,x) \sim \frac{1}{2}\sqrt{\frac{2\pi}{x}}\Big(\frac{x}{e}\Big)^x, \quad \text{as } x \to \infty.$$

3.5 Gaussian integrals

3.5.1 Error function

Let us consider the error function

$$\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$$

and the complementary error function

$$\operatorname{erfc}(x) := \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt.$$

We want to obtain an asymptotic expansion of $\operatorname{erfc}(x)$ as $x \to \infty$. Again we use the integration by parts technique. By successive integration by parts we obtain for $n \geqslant 1$,

$$\begin{aligned}
\operatorname{erfc}(x) &= \frac{2}{\sqrt{\pi}}\int_x^\infty \Big(-\frac{1}{2t}\Big)(e^{-t^2})'dt \\
&= \frac{2}{\sqrt{\pi}}\frac{e^{-x^2}}{x} - \frac{2}{\sqrt{\pi}}\int_x^\infty \frac{e^{-t^2}}{2t^2}dt \\
&= \dots \\
&= \frac{e^{-x^2}}{\sqrt{\pi}x}\sum_{k=0}^{n-1}(-1)^k(2k-1)!!\frac{1}{2^k x^{2k}} + (-1)^n\frac{2(2n-1)!!}{\sqrt{\pi}}\int_x^\infty \frac{e^{-t^2}}{(2t^2)^n}dt
\end{aligned}$$

with the convention $0!! = 1$. To show that the last expression can lead to an asymptotic expansion of $\operatorname{erfc}(x)$, we show that the reminder after n terms

$$R_n(x) = (-1)^n \frac{2(2n-1)!!}{\sqrt{\pi}} \int_x^\infty \frac{e^{-t^2}}{(2t^2)^n} dt$$

is dominated by the $(n+1)$-th term in the above sum. We use the following trick again

$$\begin{aligned}
|R_n(x)| &= \frac{2(2n-1)!!}{\sqrt{\pi}} \int_x^\infty \frac{e^{-t^2}}{(2t^2)^n} dt \\
&= \frac{2(2n-1)!!}{\sqrt{\pi}} \int_x^\infty \Big(-\frac{1}{2t}\Big) \frac{(e^{-t^2})'}{(2t^2)^n} dt \\
&\leqslant \frac{2(2n-1)!!}{\sqrt{\pi}} \frac{e^{-x^2}}{2x(2x^2)^n} \\
&= O\Big(\frac{e^{-x^2}}{x^{2n+1}}\Big)
\end{aligned}$$

as $x \to \infty$. Finally, we deduce that

$$\operatorname{erfc}(x) \sim \frac{e^{-x^2}}{\sqrt{\pi}x} \sum_{n=0}^\infty (-1)^n \frac{(2n-1)!!}{(2x^2)^n}$$

as $x \to \infty$, with the convention $(-1)!! = 1$. Observe that the last series diverges for every $x \in \mathbb{R}$!

3.5.2 "Perturbed" gaussian integral

Let us now consider the following "perturbed" gaussian integral for $a > 0$ and $\varepsilon \geqslant 0$,

$$I(a,\varepsilon) := \int_{-\infty}^\infty \exp\Big\{ -\frac{1}{2}ax^2 - \varepsilon x^4\Big\} dx$$

For $\varepsilon = 0$ we obtain the standard gaussian integral

$$I(a,0) = \sqrt{\frac{2\pi}{a}}.$$

Let us obtain now an asymptotic expansion of $I(a,\varepsilon)$ when $\varepsilon \to 0^+$. First, we proceed formally. Using the Taylor series of the exponential we obtain

$$\exp\Big\{ -\frac{1}{2}ax^2 - \varepsilon x^4\Big\} = \exp\Big\{ -\frac{1}{2}ax^2\Big\}\Big[1 - \varepsilon x^4 + \frac{1}{2!}\varepsilon^2 x^8 + \cdots + \frac{(-1)^n}{n!}\varepsilon^n x^{4n} + \dots\Big]$$

Integrating term-by-term the result (this is not justified here!), we obtain

$$I(a,\varepsilon) = \sqrt{\frac{2\pi}{a}}\Big[1 - \varepsilon m_4 + \cdots + \frac{(-1)^n}{n!}\varepsilon^n m_{4n} + \dots\Big]$$

where

$$m_{4n} = \frac{\int_{-\infty}^{\infty} x^{4n} \exp\big\{-\frac{1}{2}ax^2\big\}dx}{\int_{-\infty}^{\infty} \exp\big\{-\frac{1}{2}ax^2\big\}dx}.$$

It is well known that for all $n \geqslant 1$, we have $m_{4n} = \frac{(4n-1)!!}{a^{2n}}$. Thus, we deduce that, as $\varepsilon \to 0^+$,

$$I(a,\varepsilon) \sim \sqrt{\frac{2\pi}{a}} \sum_{n=0}^{\infty} \frac{(-1)^n(4n-1)!!}{n!a^{2n}}\varepsilon^n =: \sqrt{\frac{2\pi}{a}} \sum_{n=0}^{\infty} a_n\varepsilon^n. \tag{12}$$

By the ratio test, the radius of convergence of this series is 0, thus for all $\varepsilon > 0$ the series is divergent. This could have been anticipated by the fact that the integral $I(a,\varepsilon)$ is divergent when $\varepsilon < 0$. Next, we have to check the affirmation (12). For this, we will prove that for all $n \geqslant 0$,

$$\Big|I(a,\varepsilon) - \sqrt{\frac{2\pi}{a}} \sum_{k=0}^{n} a_k\varepsilon^k\Big| \leqslant \sqrt{\frac{2\pi}{a}}|a_{n+1}|\varepsilon^{n+1} = \sqrt{\frac{2\pi}{a}}\frac{m_{4(n+1)}}{(n+1)!}\varepsilon^{n+1}.$$

Taylor's formula implies for $y \geqslant 0$ and $n \geqslant 0$,

$$e^{-y} = 1 - y + \frac{1}{2!}y^2 + \cdots + \frac{(-1)^n}{n!}y^n + \frac{(-1)^{n+1}}{(n+1)!}e^{-\eta}y^{n+1}$$

for some $\eta \in [0,y]$. Replacing y by εx^4 in this equation and estimating the rest we obtain

$$e^{-\varepsilon x^4} = 1 - \varepsilon x^4 + \frac{1}{2!}\varepsilon^2x^8 + \cdots + \frac{(-1)^n}{n!}\varepsilon^n x^{4n} + r_{n+1}(x,\varepsilon)$$

where

$$|r_{n+1}(x,\varepsilon)| \leqslant \frac{x^{4(n+1)}}{(n+1)!}\varepsilon^{n+1}.$$

We deduce that

$$\sqrt{\frac{a}{2\pi}}I(a,\varepsilon) = \sum_{k=1}^{n} a_k\varepsilon^k + \int_{-\infty}^{\infty} r_{n+1}(x,\varepsilon)e^{-\frac{ax^2}{2}}dx.$$

Finally, it follows that

$$\begin{aligned}\Big|I(a,\varepsilon)-\sqrt{\frac{2\pi}{a}}\sum_{k=0}^{n}a_k\varepsilon^k\Big| &\leqslant \sqrt{\frac{2\pi}{a}}\Big|\int_{-\infty}^{\infty} r_{n+1}(x,\varepsilon)e^{-\frac{ax^2}{2}}dx\Big|\\ &\leqslant \sqrt{\frac{2\pi}{a}}\int_{-\infty}^{\infty}|r_{n+1}(x,\varepsilon)|e^{-\frac{ax^2}{2}}dx\\ &\leqslant \sqrt{\frac{2\pi}{a}}\frac{m_{4(n+1)}}{(n+1)!}\varepsilon^{n+1}\end{aligned}$$

which proves the result.

3.5.3 Positive gaussian integral

We study here the behaviour of

$$I(x)=\int_0^x e^{t^2}dt,$$

as $x\to\infty$. In this case case, we cannot write $I(x)=\int_0^\infty e^{t^2}dt-\int_x^\infty e^{t^2}dt$ because the right-hand side integrals have the form $\infty-\infty$. We cannot either integrate by parts directly since

$$I(x)=\left[\frac{1}{2t}e^{t^2}\right]_0^x+\frac{1}{2}\int_0^x\frac{1}{t^2}e^{t^2}dt$$

is also of the form $\infty-\infty$.

To obtain a correct asymptotic expansion of this integral, the idea is to introduce a cutoff parameter a and write

$$I(x)=\int_0^a e^{t^2}dt+\int_a^x e^{t^2}dt$$

for some fixed $0<a<x$. We can show that for fixed a, the full asymptotic expansion of $I(x)$ is independent of the first integral in the right-hand term of the above equation and is also independent of a. Then, we can use integration by parts to obtain an asymptotic expansion of the second integral in the right-hand term of the above equation. We leave as an exercise to the reader to show that

$$I(x)\sim\frac{e^{x^2}}{2x}\sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2x^2)^n},\ \text{as } x\to\infty,$$

with the convention $(-1)!!=1$.

3.6 Bivariate normal law

In this section we want to obtain an asymptotic expansion of order 2, as $\rho \to 0$, of the probability

$$P[X_1 > 0, X_2 > 0],$$

where (X_1, X_2) has bivariate normal distribution with $X_1, X_2 \sim N(0,1)$ and $\mathrm{Cov}(X_1, X_2) = \rho$. We start writing

$$P[X_1 > 0, X_2 > 0] = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_0^\infty \int_0^\infty \exp\Big\{ -\frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{2(1-\rho^2)} \Big\} dx_1 dx_2.$$

Then, we use the Taylor series of the exponential,

$$\exp\Big\{ \frac{\rho x_1 x_2}{(1-\rho^2)} \Big\} = \sum_{n=0}^{\infty} \frac{\rho^n x_1^n x_2^n}{n!(1-\rho^2)^n}$$

to deduce that

$$\begin{aligned} &P[X_1 > 0, X_2 > 0] \\ &\quad = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_0^\infty \int_0^\infty \exp\Big\{ -\frac{x_1^2 + x_2^2}{2(1-\rho^2)} \Big\} \sum_{n=0}^{\infty} \frac{\rho^n x_1^n x_2^n}{n!(1-\rho^2)^n} dx_1 dx_2. \end{aligned}$$

Permuting the sum and the integrals (by Fubini's theorem) and making the change of variables $u_i = x_i/\sqrt{1-\rho^2}$, $i = 1, 2$, we obtain that

$$P[X_1 > 0, X_2 > 0] = \sqrt{1-\rho^2} \sum_{n=0}^{\infty} \frac{m_n^2 \rho^n}{n!},$$

where $m_n := \frac{E[|Z|^n]}{2}$ and $Z \sim N(0,1)$. It is well known that, for $n \geqslant 0$,

$$E[|Z|^n] = \frac{2^{\frac{n}{2}} \Gamma(\frac{n+1}{2})}{\sqrt{\pi}}.$$

Thus, we have

$$P[X_1 > 0, X_2 > 0] = \frac{\sqrt{1-\rho^2}}{4\pi} \sum_{n=0}^{\infty} \frac{(2\rho)^n \Gamma(\frac{n+1}{2})^2}{n!}.$$

Finally, we deduce that as $\rho \to 0$,

$$P[X_1 > 0, X_2 > 0] \sim \frac{1}{4} + \frac{\rho}{2\pi}.$$

4 Appendix

4.1 Fresnel integrals

In this section we show that for $k \in \mathbb{R}^*$ and $p \in \mathbb{N} - \{1\}$,

$$I(k,p) := \int_0^\infty \exp\{iks^p\} ds = e^{\mathrm{sgn}(k)\frac{i\pi}{2p}} |k|^{-\frac{1}{p}} \frac{\Gamma(1/p)}{p}.$$

We use integration in the complex plane. We consider only the case $k > 0$. The case $k < 0$ can be treated in a similar way and is left as an exercise.

Taking $f(z) = e^{-kz^p}$ and $R > 0$, we have by Cauchy's theorem,

$$0 = \oint_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \int_{\gamma_3} f(z)dz,$$

where $\gamma = \gamma_1 + \gamma_2 + \gamma_3$ is the closed path oriented clockwise defined by $\gamma_1 = \overrightarrow{[0,R]}$, $\gamma_2 = R\exp\{i\overleftarrow{[-\pi/(2p),0]}\}$ and $\gamma_3 = \overleftarrow{[0,R]}e^{-i\frac{\pi}{2p}}$. Thus, we obtain that

$$e^{-i\frac{\pi}{2p}} \int_0^R e^{ikt^p} dt = \int_0^R e^{-kt^p} dt + \int_0^{-\frac{\pi}{2p}} e^{-kR^p e^{ip\theta}} iRe^{i\theta} d\theta.$$

By Jordan's lemma, we obtain that the second integral of the right-hand term of the above equality vanishes as $R \to \infty$. Letting $R \to \infty$, this gives us

$$e^{-i\frac{\pi}{2p}} I(k,p) = \int_0^\infty e^{-kt^p} dt.$$

Finally, making the change of variable $s = kt^p$ in the last integral and using the definition of the Γ function, we obtain

$$e^{-i\frac{\pi}{2p}} I(k,p) = k^{-\frac{1}{p}} \frac{\Gamma(1/p)}{p}.$$

This concludes the proof when $k > 0$.

4.2 Bernoulli numbers and polynomials

Many things can be said about Bernoulli numbers and polynomials. In this section, we just give a quick insight on this topic.

The Bernoulli polynomials are the elements of the unique sequence of polynomials $(B_n)_{n \geqslant 0}$ such that

- $B_0 \equiv 1$;
- $B'_{n+1} = (n+1)B_n$, for $n \geqslant 0$;
- $\int_0^1 B_n(x)dx = 0$, for $n \geqslant 1$.

The first elements of the sequence $(B_n)_{n\geqslant 0}$ are: $B_0(x) = 1$, $B_1(x) = x - \frac{1}{2}$, $B_2(x) = x^2 - x + \frac{1}{6}$, $B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$,...

The Bernoulli numbers $(b_n)_{n\geqslant 0}$ can be defined as $b_n = B_n(0)$, for all $n \geqslant 0$. The first Bernoulli numbers are $b_0 = 1$, $b_1 = -\frac{1}{2}$, $b_2 = \frac{1}{6}$, $b_3 = 0$,... Generally, it can be shown that $b_n = 0$, for even $n > 1$.

We finally recall that for all $n \geqslant 0$,

$$|b_{2n}| = \max_{x\in[0,1]} |B_{2n}(x)|$$

and

$$\frac{|b_{2n}|}{(2n)!} \leqslant \frac{2\zeta(2)}{(2\pi)^{2n}} \leqslant \frac{4}{(2\pi)^{2n}}.$$

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