Semigroups of Operators

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1 Some practical results

1.1 Elementary results

We consider $\{T_t, t \geq 0\}$ a strongly continuous semigroup in a Banach space $\mathbb{X}$. We denote by A its infinitesimal generator and by $\mathcal{D}(A)$ its domain. We remind that A is a closed operator and that $\mathcal{D}(A)$ is dense in $\mathbb{X}$. We have the following important properties:

- Let $x \in \mathbb{X}$, we have for every $t > 0$,
$$\int_0^t T_s x ds \in \mathcal{D}(A)$$
and
$$A \int_0^t T_s x ds = T_t x - x.$$

- If x belongs to $\mathcal{D}(A)$ so does $T_t x$. Furthermore, the function $t \to T_t x$ is continuously differentiable in $\mathbb{R}_+$ and for $t \geq 0$,
$$\frac{dT_t x}{dt} = AT_t x = T_t Ax \tag{1}$$

- Let $\lambda \in \mathbb{R}$ be given. We define $S_t = e^{-\lambda t} T_t$. Then $\{S_t, t \geq 0\}$ is a strongly continuous semigroup. Denote B its infinitesimal generator. We have $\mathcal{D}(B) = \mathcal{D}(A)$ and for all $x \in \mathcal{D}(B)$,
$$Bx = Ax - x$$

1.2 An important case

Let $B \in \mathcal{L}(\mathbb{X})$ be a bounded linear operator, and let $T_t = e^{tB}, t \geq 0$.
$\{T_t, t \geq 0\}$ is a semigroup of operators in $\mathcal{L}(\mathbb{X})$. Moreover, we claim that

$$\lim_{t \to 0} \|T_t - I\|_{\mathcal{L}(\mathbb{X})} = 0$$

and

$$\lim_{t \to 0} \left\| \frac{T_t - I}{t} - B \right\|_{\mathcal{L}(\mathbb{X})} = 0.$$

Note that in this case the convergence is in the operator norm!

1.3 Isomorphic semigroups

Let $\mathbb{X}$ and $\mathbb{Y}$ be two Banach spaces and let $J\colon \mathbb{X} \to \mathbb{Y}$ be an isomorphism of $\mathbb{X}$ and $\mathbb{Y}$. Suppose that $\{S_t, t \geq 0\}$ is a strongly continuous semigroup of operators in $\mathbb{X}$, with generator B. Then $\{U_t, t \geq 0\}$, where $U_t = JS_tJ^{-1}$, is a strongly continuous semigroup of operators in $\mathbb{Y}$ and its generator C equals $C = JBJ^{-1}$. To be more specific: $y \in \mathcal{D}(C)$ iff $J^{-1}y \in \mathcal{D}(B)$, and $Cy = JBJ^{-1}y$.

2 Laplace transform

We consider $\{T_t, t \geq 0\}$ a strongly continuous semigroup in a Banach space $\mathbb{X}$. Then there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\|T_t\| \leq Me^{\omega t}. \tag{2}$$

Definition 2.1 *For $\lambda > \omega$ and for all $x \in \mathbb{X}$, we define the Laplace transform of the semigroup $\{T_t, t \geq 0\}$ as*

$$R_\lambda x := \int_0^\infty e^{-\lambda s} T_s x ds.$$

Note that R_λ, for $\lambda > \omega$, are bounded operators and

$$\|R_\lambda\| \leq \frac{M}{\lambda - \omega}.$$

Fix $\lambda > \omega$. An element $x \in \mathbb{X}$ belongs to $\mathcal{D}(A)$ iff there exists a $y \in \mathbb{X}$ such that $x = R_\lambda y$. In other words x belongs to the range of the operator R_λ. Thus, we have $\mathcal{D}(A) = Range(R_\lambda)$.

2.1 A property of the Laplace transform

Let $\{T_t, t \geq 0\}$ and $\{S_t, t \geq 0\}$ two families of operators continuous in t such that

$$\|T_t\| \vee \|S_t\| \leq Me^{\omega t}$$

We can define a convolution product of the families T_t and S_t as

$$T * S(t) = \int_0^t T(t-s)S(s)ds \tag{3}$$

Then for $\lambda > \omega$, we have $\mathcal{L}[T * S] = \mathcal{L}[T]\mathcal{L}[S]$. Be careful, the convolution product of two families of operators is not commutative!

2.2 Resolvent of the generator

Fix $\lambda > \omega$. An element $x \in \mathbb{X}$ belongs to $\mathcal{D}(A)$ iff there exists a $y \in \mathbb{X}$ such that $x = R_\lambda y$. Moreover,

$$\begin{aligned} R_\lambda(\lambda - A)x &= x, \qquad x \in \mathcal{D}(A) \\ (\lambda - A)R_\lambda y &= y, \qquad y \in \mathbb{X}. \end{aligned}$$

We can easily prove the first equation by using the Laplace transform. Of course all the steps have to be properly shown. Consider equation (1) and take the Laplace transform of both sides. Noting $\mathcal{L}$ the Laplace transform we obtain

$$\begin{aligned} \mathcal{L}\frac{dT_t x}{dt} &= \mathcal{L}AT_t x \\ \lambda\mathcal{L}T_t x - x &= A\mathcal{L}T_t x \\ \lambda R_\lambda x - x &= AR_\lambda x \end{aligned}$$

which implies that $R_\lambda = (\lambda - A)^{-1}$.

As a consequence, the Laplace transform of a semigroup satisfies the Hilbert equation,

$$(\lambda - \mu)R_\lambda R_\mu = R_\mu - R_\lambda, \qquad \lambda, \mu > \omega.$$

Application of the Hilbert equation: the function $\lambda \to R_\lambda$ is continuous and also infinitely differentiable with $\frac{d^n}{d\lambda^n}R_\lambda = (-1)^n n! R_\lambda^{n+1}$.

2.3 The Cauchy problem

Let A be the infinitesimal generator of a strongly continuous semigroup $\{T_t, t \geq 0\}$. The Cauchy problem

$$\frac{dx_t}{dt} = Ax_t, \quad t \geq 0, \quad x_0 = x \in \mathcal{D}(A)$$

where x_t is a sought-for differentiable function with values in $\mathcal{D}(A)$, has the unique solution $x_t = T_t x$.

2.4 The representation of $L^1(\mathbb{R}_+)$ in $\mathcal{L}(\mathbb{X})$ related to a bounded semigroup

Let $\{T_t, t \geq 0\}$ be a stonglycontinuous semigroup of equibounded poerators, i.e. (2) is satisfied with $\omega = 0$. Moreover let $\phi \in L^1(\mathbb{R}_+)$. For any $x \in \mathbb{X}$ we can define the improper integral

$$H(\phi)x = \int_0^\infty \phi(t)T(t)xdt$$

and we obtain

$$\|H(\phi)x\| \leq M\|\phi\|_{L^1(\mathbb{R}_+)}\|x\|_{\mathbb{X}}.$$

For ϕ fixed, $H(\phi)$ is a bounded linear operator in $\mathbb{X}$. On the other hand, if we consider the application $\phi \to H(\phi)$ from $L^1(\mathbb{R}_+)$ to $\mathcal{L}(\mathbb{X})$. This application is a representation of $L^1(\mathbb{R}_+)$ in $\mathcal{L}(\mathbb{X})$ since it is an homomorphism. Indeed, we shall prove that $H(\phi * \psi) = H(\phi)H(\psi)$ for all ϕ, ψ in $L^1(\mathbb{R}_+)$.

3 Hille-Yosida Theorem

3.1 Statement of the theorem

Let $\mathbb{X}$ be a Banach space. An operator A: $\mathcal{D}(A) \to \mathbb{X}$ is the generator of a strongly continuous semigroup $\{T_t, t \geq 0\}$ in $\mathbb{X}$ such that

$$\|T_t\| \leq Me^{\omega t}, \quad t \geq 0$$

for some $M \geq 1$ and $\omega \in \mathbb{R}$ iff the following three conditions hold.
(a) A is closed and densely defined.
(b) All $\lambda > \omega$ belong to the resolvant set of A: this means that for all $\lambda > \omega$ there exists a bounded linear operator $R_\lambda = (\lambda - A)^{-1} \in \mathcal{L}(\mathbb{X})$, i.e. the unique operator such that $(\lambda - A)R_\lambda x = x$, $x \in \mathbb{X}$ and $R_\lambda(\lambda - A)x = x$, $x \in \mathcal{D}(A)$.
(c) For all $\lambda > \omega$ and all $n \geq 1$,

$$\|R_\lambda^n\| \leq \frac{M}{(\lambda - \omega)^n}.$$

Idea of proof: The difficult part is to prove that conditions (a)-(c) are sufficient. First, observe that without loss of generality we can consider the case

$\omega = 0$. Then, the key step is to introduce the family of bounded operators

$$A_\lambda = \lambda(\lambda R_\lambda - I), \quad \lambda > 0$$

called the Yosida approximation. As the operators A_λ are bounded, for each $\lambda > 0$, we can define the strongly continuous semigroup $\{e^{A_\lambda t}, t \geq 0\}$. The main idea of the proof is to show that when $\lambda \to \infty$, the semigroup $\{e^{A_\lambda t}, t \geq 0\}$ tends to a limit $\{T_t, t \geq 0\}$ which is a strongly continuous semigroup, the generator of which is A.

3.2 The Phillips perturbation Theorem

Suppose that A is the generator of a strongly continuous semigroup $\{T_t, t \geq 0\}$ satisfying (2), and B is a bounded linear operator. Then $A + B$ with domain $\mathcal{D}(A)$ is the generator of a stongly continuous semigroup $\{S_t, t \geq 0\}$ such that

$$\|S_t\| \leq Me^{(\omega + M\|B\|)t}.$$

Application: Showing existence of new processes by perturbing the original one.

3.3 Approximation theorems

Theorem 3.1 *The Trotter-Kato Theorem. Let $\{T_n(t), t \geq 0\}$, $n \geq 1$, be a sequence of strongly continuous semigroups with generators A_n. Suppose, furthermore, that there exists an $M > 0$ such that $\|T_n(t)\| \leq M$ and let $R_{\lambda,n} = (\lambda - A_n)^{-1}$, $\lambda > 0$, $n \geq 1$, denote the resolvants of A_n. If the limit*

$$R_\lambda = \lim_{n\to\infty} R_{\lambda,n} \tag{4}$$

exists in the strong topology for some $\lambda > 0$, then it exists for all $\lambda > 0$. Moreover, in such a case, there exists the strongly continuous semigroup $\{T(t), t \geq 0\}$

$$T(t)x := \lim_{n\to\infty} T_n(t)x, \quad x \in \mathbb{X}' \tag{5}$$

of operators in $\mathbb{X}' = cl(Range R_\lambda)$. The definition of $\mathbb{X}'$ does not depend on the choice of $\lambda > 0$, convergence in (5) is uniform in compact subintervals of $\mathbb{R}_+$ and we have $R_\lambda x = \int_0^\infty e^{-\lambda t} T(t)x dt$, $\lambda > 0$, $x \in \mathbb{X}'$ and $\|T(t)\|_{\mathcal{L}(\mathbb{X}')} \leq M$.

Obs: In general, there is no closed linear opeartor A such that $(\lambda - A)^{-1}$ equals R_λ, $\lambda > 0$, the limit pseudo-resolvant in the Trotter-Kato Theorem. The point is that $R_\lambda, \lambda > 0$, are in general, not injective. However, $\mathbb{X}' \cap KerR_\lambda = \{0\}$ so that $R_\lambda, \lambda > 0$ restricted to $\mathbb{X}'$ are injective.

Theorem 3.2 *The Sova-Kurtz version of the Trotter-Kato Theorem. Let $\mathbb{X}$ be a Banach space. Suppose that $\{T_n(t), t \geq 0\}$, $n \geq 1$, is a sequence of strongly continuous semigroups with generators A_n, and that there exists an $M > 0$ such that $\|T_n(t)\| \leq M$. Also, suppose that for some $\lambda > 0$ the set of y that can be expressed as $\lambda x - A_{ex}x$, where A_{ex} is the extended limit of A_n, $n \geq 1$, is dense in $\mathbb{X}$. Then, the limit (4) exists for all $\lambda > 0$. Moreover, $\mathbb{X}' = cl(\mathcal{D}(A_{ex})$ and the part A_p of A_{ex} in $\mathbb{X}'$ is single-valued and is the infinitesimal generator of the semigroup defined by (5).*

4 Lévy processes

A stochastic process $X = (X_t)_{t \geq 0}$ is called a Lévy process if the three following properties are verified:
(i) $X_0 = 0$,
(ii) the trajectories of X are càd-làg,
(iii) X has stationary and independent increments.

4.1 Convolution semigroups

A family $\{\mu_t, t \geq 0\}$ of Borel measures on $\mathbb{R}$ is said to be a convolution semigroup of measures iff
(a) $\mu_0 = \delta_0$,
(b) μ_t converges weakly to δ_0 as $t \to 0^+$,
(c) $\mu_t * \mu_s = \mu_{t+s}$, $t, s \geq 0$.

In fact, the distributions μ_t, $t \geq 0$ of a Lévy process X_t, $t \geq 0$ form a convolution semigroup.

4.2 The form of the generator of a convolution semigroup

Let μ_t, $t \geq 0$ be a convolution semigroup on $\mathbb{R}$. Define the semigroup $\{T_t, t \geq 0\}$ in $\mathbb{X} = BUC(\mathbb{R})$ by $T_t = T_{\mu_t}$. This semigroup is strongly continuous in

$\mathbb{X}$. Let us call A its generator. In general, finding an explicit form for A is difficult, if possible at all. In fact, the domain of A contains the space $\mathbb{X}_2$ of all twice differentiable functions in $\mathbb{X}$ with both derivatives in $\mathbb{X}$. A restricted to $\mathbb{X}_2$ can be described in more detail.

Theorem 4.1 *Let μ_t, $t \geq 0$ be a convolution semigroup and let A be the generator of the corresponding semigroup $\{T_t, t \geq 0\}$ in $\mathbb{X} = BUC(\mathbb{R})$. Then $\mathbb{X}_2 \subset \mathcal{D}(A)$. Moreover, there exists an $a \in \mathbb{R}$ and a finite Borel measure m on $\mathbb{R}$ such that*

$$Ax(\sigma) = ax'(\sigma) + \int_{\mathbb{R}} [x(\tau + \sigma) - x(\sigma) - x'(\sigma)y(\tau)] \frac{\tau^2 + 1}{\tau^2} m(d\tau), \qquad (6)$$

$x \in \mathbb{X}_2$, where $y(\tau) = \frac{\tau}{\tau^2+1}$.

Corollary 4.1 *The set $\mathbb{X}_2$ is a core for A. In particular, A is fully determined by (6).*

We end this subsection by giving the Lévy-Khintchine formula

Theorem 4.2 *With a and m given above, we have*

$$\int_{\mathbb{R}} e^{i\tau\xi} \mu_t(d\tau) = \exp\Big\{it\xi a + t \int_{\mathbb{R}} \Big(e^{i\xi\tau} - 1 - \frac{i\xi\tau}{\tau^2 + 1}\Big) \frac{\tau^2 + 1}{\tau^2} m(d\tau)\Big\}.$$

5 Markov processes

5.1 Definition

Definition 5.1 *Let $(\Omega, \mathcal{F}, P)$ be a probability space. A process $X_t, t \geq 0$ on $(\Omega, \mathcal{F}, P)$, with values in a topological space S, is said to be a Markov process if for every $t \geq 0$, the σ-algebra $\mathcal{F}_t = \sigma\{X_s, s \geq t\}$ depends on $\sigma\{X_s, s \leq t\}$ only through $\sigma(X_t)$. In other words, for all $A \in \sigma\{X_s, s \geq t\}$ and $B \in \sigma\{X_s, s \leq t\}$,*

$$P[A \cap B \mid X_t] = P[A \mid X_t] P[B \mid X_t]$$

for all $t > 0$.

We can check that the definition above is equivalent to each one of the following conditions

- for every n and $t \leq t_1 \leq \cdots \leq t_n$ and Borel sets $B_i, i = 1, \ldots, n$,

$$P[X(t_i) \in B_i, i = 1, \ldots, n \mid \mathcal{F}_t] = P[X(t_i) \in B_i, i = 1, \ldots n \mid X_t],$$

- $P[X(s) \in B \mid \mathcal{F}_t] = P[X(s) \in B \mid X_t]$, $s \geq t$, $B \in \mathcal{B}(S)$,
- $E[f(X(s)) \mid \mathcal{F}(t)] = E[f(X(s)) \mid X(t)]$ for any $f \in BM(S)$.

Obs: If S is a metric space then the conditions above are also equivalent to

$$E[f(X(s)) \mid \mathcal{F}(t)] = E[f(X(s)) \mid X(t)] \text{ for any } f \in BC(S).$$

If $S = \mathbb{R}$, the conditions above are also equivalent to

$$E[f(X(s)) \mid \mathcal{F}(t)] = E[f(X(s)) \mid X(t)] \text{ for any } f \in C_c(\mathbb{R}).$$

5.2 Transition kernels of time-homogeneous Markov processes

Definition 5.2 *The transition kernel of a (time-homogeneous) Markov process is a function $K(t, \tau, B)$ of three variables $t \geq 0$, $p \in S$, $B \in \mathcal{B}(S)$ which satisfies the following properties.*
(a) $K(t, p, \cdot)$ is a probability measure on $(S, \mathcal{B}(S))$, for all $t \geq 0$, $p \in S$.
(b) $K(0, p, \cdot) = \delta_p$.
(c) $K(t, \cdot, B)$ is measurable for all $t \geq 0$ and $B \in \mathcal{B}(S)$.
(d) The Chapman-Kolmogorov equation is satisfied:

$$\int_S K(s, q, B) K(t, p, dq) = K(t + s, p, B).$$

A family $\{X_t, t \geq 0\}$ of random variables on a probability space $(\Omega, \mathcal{F}, P)$ with values in S is a Markov process with transition kernel K if for $t > s$:

$$P[X(t) \in B \mid \mathcal{F}_s] = P[X(t) \in B \mid X(s)] = K(t - s, X(s), B), \quad B \in \mathcal{B}(S).$$

In other words, for every $s \geq 0$, it exists a regular version of the conditional expectation $E[\ \cdot \mid X(s)]$. The measure $K(t, p, \cdot)$ is the distribution of the position of the process at time t given that at time zero it started at p.

5.3 Semigroups of operators related to transition kernels of Markov processes

With a transition kernel one may associate a semigroups of non-negative operators in $BM(S)$ by

$$T_t x(p) = \int_S x(q) K(t, p, dq), \quad t \geq 0. \tag{7}$$

Clearly, $\|T_t\| = 1$ and $T_t 1_S = 1_S$, $t \geq 0$.

Theorem 5.1 *Let S be a compact space and $\{T_t, t \geq 0\}$ be a semigroup of non-negative contraction operators in $C(S)$ such that $T_t 1_S = 1_S$. Then, there exists the unique transition kernel K such that (7) holds for all $x \in C(S)$.*

Theorem 5.2 *Let S be a locally compact space (but not compact) and let S_Δ be the one-point compactification of S. Let $\{T_t, t \geq 0\}$ be a semigroup of non-negative contraction operators in $C_0(S)$. Then, there exists the unique transition kernel K on S_Δ such that (7) holds for all $x \in C_0(S)$, and $K(t, \Delta, \cdot) = \delta_\Delta$.*

The point Δ is called cemetery. In general $K(t, p, S) \leq 1$, $p \in S$ as $K(t, p, \{\Delta\}) \geq 0$.

5.4 Feller semigroups

Definition 5.3 *Consider the semigroup $\{T_t, t \geq 0\}$ introduced in (7). If S is locally compact and the semigroup $\{T_t, t \geq 0\}$ leaves $C_0(S)$ invariant and is non-negative and strongly continuous restricted to this subspace, we say that $\{T_t, t \geq 0\}$ is a Feller semigroup and that the related process is a Feller process. As an example, Lévy processes are Feller.*

5.4.1 Generators of Feller processes

Consider a semigroup of the form (7) given above. If S is locally compact and the semigroup leaves $C_0(S)$ invariant and is strongly continuous as restricted to this subspace, we say that $\{T_t, t \geq 0\}$ is a Feller semigroup and that the related kernel is a Feller kernel. We note that Lévy processes are Feller processes.

The reason for defining Feller semigroups is that these semigroups present a certain regularity. This means that the processes associated present some important properties such as the strong Markov property, càd-làg paths...

Definition 5.4 *Let S be a locally compact space. An operator A: $C_0(S) \supset \mathcal{D}(A) \to C_0(S)$ is said to satisfy the* **positive maximum principle** *if for any $x \in \mathcal{D}(A)$ and $p \in S$, $x(p) = \sup_{q \in S} x(q) \geq 0$ implies $Ax(p) \leq 0$.*

5.4.2 Generators of Feller processes

Theorem 5.3 *Let S be a locally compact space. An operator A in $C_0(S)$ is the generator of a semigroup related to a Feller kernel iff*
(a) $\mathcal{D}(A)$ is dense in $C_0(S)$,
(b) A satisfies the positive maximum principle,
(c) for some $\lambda_0 > 0$, the range of the operator $\lambda_0 - A$ equals $C_0(S)$.

Obs: Operators satisfying the positive maximum principle are dissipative. A linear operator A: $\mathbb{X} \supset \mathcal{D}(A) \to \mathbb{X}$ is said to be **dissipative** if for all $x \in \mathcal{D}(A)$ and $\lambda > 0$, $\|\lambda x - Ax\| \geq \lambda \|x\|$.

Sometimes it is convenient to have the following version of Theorem 5.3.

Theorem 5.4 *Let S be a locally compact space. An operator A in $C_0(S)$ is the generator of a semigroup related to a Feller kernel iff*
(a) $\mathcal{D}(A)$ is dense in $C_0(S)$,
(b) if $x \in \mathcal{D}(A)$, $\lambda > 0$ and $y = \lambda x - Ax$ then $\lambda \inf_{p \in S} x(p) \geq \inf_{p \in S} y(p)$,
(c) for some $\lambda_0 > 0$, the range of the operator $\lambda_0 - A$ equals $C_0(S)$.

5.4.3 Pre-generators of Feller processes

The problem with applying theorems 5.3 and 5.4 is that the whole domain of an operator is rarely known explicitely, and we must be satisfied with knowing its core. Hence, we need to characterize operators which must be extented to a generator of a Feller semigroup. In particular, such operators must be **closable**

Definition 5.5 *An operator A: $\mathbb{X} \supset \mathcal{D}(A) \to \mathbb{X}$ is said to be closable if there exists a closed linear operator B such that $Bx = Ax$ for $x \in \mathcal{D}(A)$.*

We now give a characterization of closable operators.

Proposition 5.1 *Let A be a linear operator in a Banach space $\mathbb{X}$. The following conditions are equivalent:*
(a) A is closable,
(b) the closure of the graph G_A of A in the space $\mathbb{X} \times \mathbb{X}$ equiped with the norm $\|(x, y)\| = \|x\| + \|y\|$ is a graph of a closed operator,
(c) if $x_n \in \mathcal{D}(A)$, $n \geq 1$, $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} Ax_n$ exists, then $\lim_{n\to\infty} Ax_n = 0$.

Definition 5.6 *The* **closure** *$\bar{A}$ of a closable operator A is the unique closed operator such that $G_{\bar{A}} = clG_A$.*

Theorem 5.5 *Let S be a locally compact space and A be a linear operator $C_0(S) \supset \mathcal{D}(A) \to C_0(S)$. A is closable and its closure $\bar{A}$ generates a Feller semigroup iff*
(a) $\mathcal{D}(A)$ is dense in $C_0(S)$,
(b) if $x \in \mathcal{D}(A)$, $\lambda > 0$ and $y = \lambda x - Ax$ then $\lambda \inf_{p\in S} x(p) \geq \inf_{p\in S} y(p)$,
(c) for some $\lambda_0 > 0$, the range of the operator $\lambda_0 - A$ is dense in $C_0(S)$.

Theorem 5.6 *Let S be a locally compact space and A be a linear operator $C_0(S) \supset \mathcal{D}(A) \to C_0(S)$. A is closable and its closure $\bar{A}$ generates a Feller semigroup iff*
(a) $\mathcal{D}(A)$ is dense in $C_0(S)$,
(b) A satisfies the positive maximum principle,
(c) for some $\lambda_0 > 0$, the range of the operator $\lambda_0 - A$ is dense in $C_0(S)$.

6 The Feynman-Kac formula

Theorem 6.1 *The Trotter product formula. Suppose that A, B and C are generators of c_0 semigroups $\{S(t), t \geq 0\}$, $\{T(t), t \geq 0\}$ and $\{U(t), t \geq 0\}$ of contractions in a Banach space $\mathbb{X}$. Suppose that $\mathcal{D}$ is a core for C and that $\mathcal{D} \subset \mathcal{D}(A) \cap \mathcal{D}(B)$ and $Cx = Ax + Bx$ for $x \in \mathcal{D}$. Then,*

$$U(t) = \lim_{t\to\infty} \left[S\left(\frac{t}{n}\right) T\left(\frac{t}{n}\right) \right]^n, \quad t \geq 0,$$

strongly.

Application: the Feynman-Kac formula.

Let $\{X_t, t \geq 0\}$, be a Lévy process, and let $\{T(t), t \geq 0\}$ be the related semigroup in $\mathbb{X} = C_0(\mathbb{R})$ or $\mathbb{X} = C[-\infty, \infty]$. Moreover, lat A be the infinitesimal generator of $\{T(t), t \geq 0\}$ and B be the operator in $\mathbb{X}$ given by $Bx = bx$ where b is a fixed member of $\mathbb{X}$. The semigroup $\{U(t), t \geq 0\}$ generated by $A + B - \beta I$ where $\beta = \|b\|$ is given by

$$U(t)x = e^{-\beta t} E\Big[e^{\int_0^t b(\tau + X_s) ds} x(\tau + X_t)\Big].$$

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