Lista 1 – ME 501 Processos Estocásticos

1) Prove que $\{X_n\}$ uma cadeia de markov, tal que:
$P(X_n = j|X_k = l, X_m = i) = P(X_n = j|\ X_m = i), 0 < k < m < n$

Pela propriedade da CM, sabemos que

$$P(X_n = j|X_k = l, X_m = i) = \frac{P(X_n = j, X_k = l, X_m = i)}{P(X_k = l, X_m = i)} =$$

$$\frac{\sum_{i_{n-1},\dots,i_0} P(X_n = j, X_{n-1} = i_{n-1}, \dots, X_{m+i} = i_{m+1}, X_m = i, \dots, X_k = l, \dots, X_0 = i_0)}{P(X_k = l, X_m = i)} =$$

$$\frac{\sum_{i_{n-1},\dots,i_0} P(X_n = j, X_{n-1} = i_{n-1}, \dots, X_{m+i} = i_{m+1}|X_m = i, \dots, X_k = l, \dots, X_0 = i_0) \cdot P(X_m = i, \dots, X_k = l, \dots, X_0 = i_0)}{P(X_k = l, X_m = i)}$$

$$= \frac{\sum_{i_{n-1},\dots,i_0} P(X_n = j, X_{n-1} = i_{n-1}, \dots, X_{m+i} = i_{m+1}\ |X_m = i) \cdot P(X_m = i, \dots, X_k = l, \dots, X_0 = i_0)}{P(X_k = l, X_m = i)}$$

$$=^{def}\ P(X_n = j|\ X_m = i) \cdot \frac{\sum_{i_{n-1},\dots,i_0} P(X_m = i, \dots, X_k = l, \dots, X_0 = i_0)}{P(X_k = l, X_m = i)}$$

$$= P(X_n = j|\ X_m = i)$$

Logo, provamos que $\{X_n\}$ é uma cadeia de Markov.

2) $\{X_n\}$ é *uma Cadeia de Markov, pois o processo depende unicamente do último estado, antes da próxima escolha das urnas, ou seja, se estamos no estado* $\boldsymbol{k}$ *no tempo* $\boldsymbol{n}$*, então a probabi − lidade de se mover para o estado* $\boldsymbol{x}$ *no tempo* $\boldsymbol{n+1}$ *não depende de* $\boldsymbol{n}$*, e somente depende do estado* $\boldsymbol{k}$ *onde estamos.*

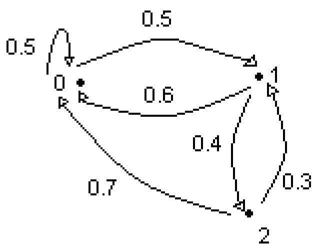
Para calcularmos a matriz de transição temos:

$X_n = \#\ bolas\ na\ urna\ 1$
$X_n \in \{0, \dots, 5\}$
Sabemos que: $\quad p_{ij} = P(X_{n+1} = j|X_n = i)$
Então,
$p_{oo} = P(X_{n+1} = 0|X_n = 0) = P(escolher\ urna\ 1|X_n = 0) = {}^1/_2$
$p_{o1} = P(X_{n+1} = 1|X_n = 0) = P(escolher\ urna\ 2|X_n = 0) = {}^1/_2$
$p_{o2} = p_{o3} = p_{o4} = p_{o5} = P(X_{n+1} = k|X_n = 0) = 0, onde\ k = 2,3,4,5$
$p_{1o} = P(X_{n+1} = 0|X_n = 1) = {}^1/_2$
$p_{11} = P(X_{n+1} = 1|X_n = 1) = 0$
$p_{12} = P(X_{n+1} = 2|X_n = 1) = {}^1/_2$

$p_{13} = p_{14} = p_{15} = \ P(X_{n+1} = k|X_n = 1) = 0, onde\ k = 3, 4, 5$
$p_{20} = P(X_{n+1} = 0|X_n = 2) = 0$
$p_{21} = P(X_{n+1} = 1|X_n = 2) = {}^1/_2$
$p_{22} = P(X_{n+1} = 2|X_n = 2) = 0$
$p_{23} = P(X_{n+1} = 3|X_n = 2) = {}^1/_2$
$p_{24} = p_{25} = P(X_{n+1} = k|X_n = 2) = 0, onde\ k = 4, 5$
$p_{30} = p_{31} = p_{33} = p_{35} = P(X_{n+1} = k|X_n = 3) = 0, onde\ k = 0, 1, 3, 5$
$p_{32} = P(X_{n+1} = 2|X_n = 3) = {}^1/_2$
$p_{34} = P(X_{n+1} = 4|X_n = 3) = {}^1/_2$
$p_{40} = p_{41} = p_{42} = p_{44} = P(X_{n+1} = k|X_n = 4) = 0, onde\ k = 0, 1, 2, 4$
$p_{43} = P(X_{n+1} = 3|X_n = 4) = {}^1/_2$
$p_{45} = P(X_{n+1} = 5|X_n = 4) = {}^1/_2$
$p_{50} = p_{51} = p_{52} = p_{53} = P(X_{n+1} = k|X_n = 5) = 0, onde\ k = 0, 1, 2, 3$
$p_{54} = P(X_{n+1} = 4|X_n = 5) = {}^1/_2$
$p_{55} = P(X_{n+1} = 5|X_n = 5) = {}^1/_2$
$Assim, nossa\ matriz\ de\ transição\ será$:

$$\begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix} = \mathcal{P}$$

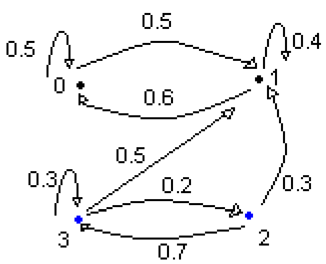
3)



a) $\mathcal{P}_1 = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.6 & 0 & 0.4 \\ 0.7 & 0.3 & 0 \end{bmatrix}$

Classes:

{0, 1,2} - recorrente irredutível



b) $\mathcal{P}_2 = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.6 & 0.5 & 0 & 0 \\ 0 & 0.3 & 0 & 0.7 \\ 0 & 0.5 & 0.2 & 0.3 \end{bmatrix}$

Classes:

{0, 1} – recorrente

{2, 3} – transiente

4) $\mathcal{P}_1 = \begin{bmatrix} 0.1 & 0.5 & 0 \\ 0.6 & 0 & 0.4 \\ 0.7 & 0.3 & 0 \end{bmatrix}$

Distribuição inicial: (0.7; 0.2; 0.1)

a) t=2

$\mathcal{P}^2 \rightarrow P(X_2 = i) = 0.7{p_{0i}}^2 + 0.2{p_{1i}}^2 + 0.1{p_{2i}}^2$

$$\mathcal{P}^2 = \mathcal{P}.\mathcal{P} = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} 0.1 & 0.5 & 0 \\ 0.6 & 0 & 0.4 \\ 0.7 & 0.3 & 0 \end{bmatrix} = \begin{bmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{bmatrix}$$

$P(X_2 = 0) = 0.7x0.43 + 0.2x0.24 + 0.1x0.36 = 0.385$

$P(X_2 = 1) = 0.7x0.31 + 0.2x0.42 + 0.1x0.35 = 0.336$

$P(X_2 = 2) = 0.7x0.26 + 0.2x0.34 + 0.1x0.29 = 0.279$

b) t = 0, 1, 2, 3

$P(X_0 = 0, X_1 = 2, X_2 = 2, X_3 = 1)$

$P(X_0 = 0)P(\ X_1 = 2|X_0 = 0)P(\ X_2 = 2|X_1 = 2, X_0 = 0)P(X_3 = 1|X_2 = 2, X_1 = 2, X_0 = 0) =$

$P(X_0 = 0)P(\ X_1 = 2|X_0 = 0)P(\ X_2 = 2|X_1 = 2)P(X_3 = 1|X_2 = 2) =$
$= 0.7x0.4x0.3x0.4 = 0.0336$

c) Distribuição estacionária

$$\begin{cases} \pi_0 = \pi_0 0.1 + \pi_1 0.6 + \pi_2 0.3 \\ \pi_1 = \pi_0 0.5 + \pi_1 0.2 + \pi_2 0.4 \\ \pi_2 = \pi_0 0.4 + \pi_1 0.2 + \pi_2 0.3 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}$$

$$\begin{cases} \pi_0 = \pi_0 0.1 + \pi_1 0.6 + \pi_2 0.3 \\ \\ \pi_1 - \pi_2 = \pi_0 0.1 + \pi_2 0.1 \\ \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}$$

$$\begin{cases} \pi_0 - \pi_1 + \pi_2 = \pi_1 0.6 + \pi_2 0.2 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}$$

$$\begin{cases} \pi_0 = \pi_1 1.6 - \pi_2 0.8 \\ \pi_0 = 1 - \pi_1 - \pi_2 \end{cases}$$

$\pi_1 1.6 - \pi_2 0.8 = 1 - \pi_1 - \pi_2$
$\pi_2(1 - 0.8) = 1 + \pi_1(-1 - 1.6)$
$\pi_2 0.2 = 1 + \pi_1 2.6$
$\boldsymbol{\pi_2} = {}^{1 - \pi_1 2.6}\!/_{0.2} = \mathbf{5 - 13}\boldsymbol{\pi_1}$
$\pi_0 = 1 - \pi_1 - 5 + 13\pi_1$
$\boldsymbol{\pi_0} = \mathbf{-4 + 12}\boldsymbol{\pi_1}$

$\pi_0 = \pi_0 0.1 + \pi_1 0.6 + \pi_2 0.3$

$12\pi_1 - 4 = 0.1(12\pi_1 - 4) + \pi_1 0.6 + 0.3(5 - 12\pi_1)$

$12\pi_1 - 4 = 1.2\pi_1 - 0.4) + \pi_1 0.6 + 1.5 - 3.9\pi_1$

$\pi_1(12 - 1.2 - 0.6 + 3.9) = 4 - 0.4 + 1.5$

$\pi_1 14.1 = 5.1$

$\boldsymbol{\pi_1} = \mathbf{0.3617}$
$\boldsymbol{\pi_0} = \mathbf{0.3404}$
$\boldsymbol{\pi_2} = \mathbf{0.2979}$

5) Para provarmos esse exercício, provaremos pelo método da indução, onde temos:

$$\mathcal{P}_1 = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

$$\mathcal{P}^2 = \begin{pmatrix} p^2 + (1-p)^2 & 2p(1-p) \\ 2p(1-p) & p^2 + (1-p)^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}(2p-1)^2 & \frac{1}{2} - \frac{1}{2}(2p-1)^2 \\ \frac{1}{2} - \frac{1}{2}(2p-1)^2 & \frac{1}{2} + \frac{1}{2}(2p-1)^2 \end{pmatrix}$$

$$\begin{pmatrix} 2p^2 - 2p + 1 & 2p - 2p^2 \\ 2p - 2p^2 & 2p^2 - 2p + 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}(4p^2 - 4p + 1) & \frac{1}{2} - \frac{1}{2}(4p^2 - 4p + 1) \\ \frac{1}{2} - \frac{1}{2}(4p^2 - 4p + 1) & \frac{1}{2} + \frac{1}{2}(4p^2 - 4p + 1) \end{pmatrix}$$

$$\begin{pmatrix} 2p^2 - 2p + 1 & 2p - 2p^2 \\ 2p - 2p^2 & 2p^2 - 2p + 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + 2p^2 - 2p + 1/2 & \frac{1}{2} - 2p^2 + 2p - 1/2 \\ \frac{1}{2} - 2p^2 + 2p - 1/2 & \frac{1}{2} + 2p^2 - 2p + 1/2 \end{pmatrix}$$

$$\begin{pmatrix} 2p^2 - 2p + 1 & 2p - 2p^2 \\ 2p - 2p^2 & 2p^2 - 2p + 1 \end{pmatrix} = \begin{pmatrix} 2p^2 - 2p + 1 & -2p^2 + 2p \\ -2p^2 + 2p & 2p^2 - 2p + 1 \end{pmatrix}$$

Logo, temos que para n=1, 2 a matriz de transição é válida.

Supomos então, que é valido n-1:

$$\mathcal{P}^{n-1} = \begin{pmatrix} \frac{1}{2}+\frac{1}{2}(2p-1)^{n-1} & \frac{1}{2}-\frac{1}{2}(2p-1)^{n-1} \\ \frac{1}{2}-\frac{1}{2}(2p-1)^{n-1} & \frac{1}{2}+\frac{1}{2}(2p-1)^{n-1} \end{pmatrix}$$

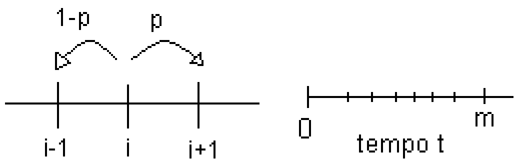
Então,

$$\mathcal{P}^{n} = \mathcal{P}^{n-1}\mathcal{P}^{1} = \begin{pmatrix} \frac{1}{2}+\frac{1}{2}(2p-1)^{n-1} & \frac{1}{2}-\frac{1}{2}(2p-1)^{n-1} \\ \frac{1}{2}-\frac{1}{2}(2p-1)^{n-1} & \frac{1}{2}+\frac{1}{2}(2p-1)^{n-1} \end{pmatrix}\begin{pmatrix} \frac{1}{2}+\frac{1}{2}(2p-1)^{1} & \frac{1}{2}-\frac{1}{2}(2p-1)^{1} \\ \frac{1}{2}-\frac{1}{2}(2p-1)^{1} & \frac{1}{2}+\frac{1}{2}(2p-1)^{1} \end{pmatrix}$$

$$\mathcal{P}^{n} = \begin{pmatrix} \frac{1}{2}+\frac{1}{2}(2p-1)^{n} & \frac{1}{2}-\frac{1}{2}(2p-1)^{n} \\ \frac{1}{2}-\frac{1}{2}(2p-1)^{n} & \frac{1}{2}+\frac{1}{2}(2p-1)^{n} \end{pmatrix}$$

Assim, concluímos que é válido para n.

6) Temos um passeio aleatório simples



E podemos considerar que é uma cadeia de markov, pois dependemos apenas do último resultado encontrado e: p, 1-p >0, p+1-p=1

Nossa cadeia de markov pode ir para frente ($\longrightarrow$) ou para trás ($\longleftarrow$)

Assim temos:

$$\left.\begin{array}{l} \text{Passos} \longrightarrow \; - \text{passos} \longleftarrow \; = m \\ \text{Passos} \longrightarrow \; + \text{passos} \longleftarrow \; = t \end{array}\right\} \text{ são independentes}$$

E o #passos $\longrightarrow$ = $\frac{t+m}{2}$, # passos $\longleftarrow$ =$\frac{t-m}{2}$

E, obtemos a seguinte probabilidade:

$$\begin{cases} P(X_t = m) = \begin{pmatrix} \frac{t+m}{2} \\ \frac{t-m}{2} \end{pmatrix} p^{\frac{(t+m)}{2}}(1-p)^{\frac{t-m}{2}}, se\ t\ e\ m\ s\tilde{a}□\ dois\ pares\ ou\ dois\ í□□□□□□ \\ 0, se\ um\ é□□□□□□□í□□□□ \end{cases}$$

7) $Temos\ p_{ij}{}^{r} > 0, \forall i, j$

$\rightarrow \forall n \geq r, \quad p_{ij}{}^{n} > 0, \forall i, j$

$Assim, consideramos\ \ \mathcal{P}^{n} = \mathcal{P}^{n-r}\mathcal{P}^{r}$

$então,$

$p_{ij}{}^{n} = \sum_{l} p_{il}{}^{n-r} p_{lj}{}^{r} > 0\ pois \sum_{l} p_{il}{}^{n-r} = 1 \rightarrow \exists l\ tal\ que\ p_{il}{}^{n-r} > 0$

$logo, \forall n > r, todos\ os\ elementos\ da\ \ \mathcal{P}^{n}\ também\ serão\ positivos.$

8) $Temos\ p_{ij}{}^{m} > 0,$

$Consideramos\ que,$

$p_{ij}{}^{m} = \sum_{l_{1,\dots,}l_{m-1}} p_{il_1} p_{l_1 l_2} \dots p_{l_{m-1} j} > 0, \exists l_1, \dots, l_{m-1}\ t.q.seja > 0$

$logo, a\ partir\ dos\ caminhos\ de\ probabilidade\ positiva,$

$se\ temos\ p_{il_1}, p_{l_1 l_2}, p_{l_3 l_1}, p_{l_1 l_5}, \dots > 0,\ onde\ l_1 = l_4,$

$teremos\ passos\ repetidos\ ao\ longo\ do\ caminho, no\ qual\ devemos\ \ descartar\ os\ passos\ retidos,$

$e\ teremos\ no\ máximo\ teremos\ \boldsymbol{m}\ passos\ ou\ menos.$

9) $Temos\ que\ f_{ij}{}^{n} = P(X_n = j, X_k \neq j, k = 1, \dots, n-1 | X_0 = i)$

$Sabemos\ também\ que\ p_{ij}{}^{n} = P(X_n = j | X_0 = i)$

$$= \sum_{k=0}^{n} P(X_n = j | X_0 = i, A_k)\, P(A_k | X_0 = i)$$

$$= \sum_{k=0}^{n} P(X_n = j | X_0 = i, X_0 \neq j, \dots, X_{k-1} \neq j, X_k = j)\, P(A_k | X_0 = i)$$

$$= \sum_{k=0}^{n} P(X_n = j | X_k = j)\, P(A_k | X_0 = i) = \sum p_{jj}^{n-k} f_{ij}^{k}$$

$Obs.: A_k = \{X_k = j, X_h \neq j, h = 0, \dots, k-1\},$

elementos disjuntos, onde consideramos ser a primeira vez em que $\mathrm{X} = \mathrm{j}$

no instante de tempo k

$f_{ij}^{k} = P(A_k)$

Priscila Coelho da Cruz

RA 083999